Math-21401: Functional Analysis

Prerequisites: Real Analysis, Linear Algebra

Specific Objectives of Course: This course extends methods of linear algebra and analysis to spaces of functions, in which the interaction between algebra and analysis allows powerful methods to be developed. The course will be mathematically sophisticated and will use ideas both from linear algebra and analysis.

Course Outlines:

Metric space: Review of metric spaces, convergence in metric spaces, complete metric spaces, dense sets and separable spaces, no-where dense sets, Baire category theorem. *Normed spaces:* Normed linear spaces, Banach spaces, Hahn-Banach theorem and its consequences, orthogonal and orthonormal systems, Gram-Schmidt process, equivalent norms, linear operator, finite dimensional normed spaces, continuous and bounded linear operators, dual spaces.

Inner product spaces: Definition and examples, orthonormal sets and basis, annihilators, projections, linear functional on Hilbert spaces, reflexivity of Hilbert spaces.

Recommended Books:

1) V. Balakrishnan, Applied Functional Analysis, 2nd Edition, Springer-Verlag, Berlin (2013)
2) J. B. Conway, A Course in Functional Analysis, 2nd Edition, Springer-Verlag, Berlin (2010)
3) K. Yosida, Functional Analysis, 5th Edition, Springer-Verlag, Berlin (2011)

E. Kreyszig, Introduction to Functional Analysis with Applications, John Wiley and Sons (2009)

Metric Spaces

Real Valued Function

Let $f: A \to R$ be a function. Clearly domain of f is A, in other words f is defined on A. Since co-domain of f is R, we can say that f is real valued function.

Metric

Let X be a non-empty set and R be a set of real numbers.

Let $d: X \times X \to R$ be a function

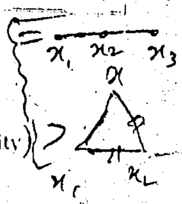
Then "d" is called "metric" on X, if "d" satisfies each of the following four conditions;

(M$_1$) $d(x_1, x_2) \geq 0 \quad \forall\, x_1, x_2 \in X$

(M$_2$) $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2 \quad \forall\, x_1, x_2 \in X$

(M$_3$) $d(x_1, x_2) = d(x_2, x_1) \quad \forall\, x_1, x_2 \in X$ (Symmetric Property)

(M$_4$) $d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3) \quad \forall\, x_1, x_2, x_3 \in X$ (Triangular Inequality)



If "d" is a "metric" on X then the pair (X, d) is called metric space.

Note

The non-negative real number $d(x_1, x_2)$ is called distance between points x_1 and x_2 in the metric "d".

Usual Metric on R

Let $d: R \times R \to R$ be a metric on R given by $d(x_1, x_2) = |x_1 - x_2|$

Then "d" is called a usual metric on R and (R, d) is called usual metric space.

Usual Metric on R^2

Let $d: R^2 \times R^2 \to R$ be a metric on R^2 given by

$d(P_1,P_2)$ $$d[(x_1, y_1), (x_2, y_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Then "d" is called a usual metric on R^2 and (R^2, d) is usual metric space.

Usual Metric on R^3

Let $d: R^3 \times R^3 \to R$ be a metric on R^3 given by

$$d[(x_1, y_1, z_1), (x_2, y_2, z_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Then "d" is called a usual metric on R^3 and (R^3, d) is usual metric space.

Note

When we say that R is a metric space without giving a metric on R then it is assumed that metric on R is "usual metric".

Similarly we take the case of R^2 and R^3.

Example

Let X be the set of all towns marked on a plane geographically map and let $d(x_1, x_2)$ be the length of the shortest rout from town x_1 to x_2. Show that "d" is a metric on X.

Solution

Here function $d: X \times X \to R$ is defined as

$d(x_1, x_2)$ =Length of shortest route from town x_1 to x_2.

(M$_1$) Since (Length of shortest route from town x_1 to x_2) ≥ 0

$\therefore \quad d(x_1, x_2) \geq 0$

(M$_2$) Let $d(x_1, x_2) = 0 \Rightarrow$ Length of shortest route from town x_1 to $x_2 = 0$

$\Rightarrow \quad x_1 = x_2$

Let $\quad x_1 = x_2 \quad \Rightarrow$ Length of shortest route from town x_1 to $x_2 = 0$

$\Rightarrow \quad d(x_1, x_2) = 0$

(M$_3$) Since $d(x_1, x_2)$ = Length of shortest route from town x_1 to x_2

= Length of shortest route from town x_2 to x_1

$= d(x_2, x_1)$

(M$_4$) Let $x_1, x_2, x_3 \in X$

Then x_1, x_2, x_3 are non-collinear or collinear

If x_1, x_2, x_3 are non-collinear, then they form a triangle and we know that sum of length of two sides of a triangle is always greater than the third side.

$\therefore \quad d(x_1, x_2) + d(x_2, x_3) > d(x_1, x_3)$ ------------ --- (i)

Let x_1, x_2, x_3 are collinear.

Then $\quad d(x_1, x_2) + d(x_2, x_3) = d(x_1, x_3)$ -------- --- (ii)

From (i) and (ii), we get

$d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3)$

Hence "d" is a metric on X.

Example

Let $X = R$ be the set of all real numbers and let $d: R \times R \to R$ be defined by $d(x_1, x_2) = |x_1 - x_2|$ denotes the absolute value of the number $x_1 - x_2$. Show that (R, d) is a metric space.

Solution

Here function $d: R \times R \to R$ is defined as

$d(x_1, x_2) = |x_1 - x_2|$

(M$_1$) Since $|x_1 - x_2| \geq 0$

$\therefore \quad d(x_1, x_2) \geq 0$

(M$_2$) Let $d(x_1, x_2) = 0 \quad \Rightarrow \quad |x_1 - x_2| = 0.$

$\Rightarrow \quad x_1 - x_2 = 0$

$\Rightarrow \quad x_1 = x_2$

Let $\quad x_1 = x_2 \quad \Rightarrow \quad x_1 - x_2 = 0$

$\Rightarrow \quad |x_1 - x_2| = 0$

$\Rightarrow \quad d(x_1, x_2) = 0$

Thus $d(x_1, x_2) = 0 \iff x_1 = x_2$

(M$_3$) Since $d(x_1, x_2) = |x_1 - x_2|$

$$= |-(x_2 - x_1)|$$
$$= |x_2 - x_1|$$
$$= d(x_2, x_1)$$

(M$_4$) Since $d(x_1, x_2) = |x_1 - x_2|$

$$d(x_2, x_3) = |x_2 - x_3|$$
$$d(x_1, x_3) = |x_1 - x_3|$$

Now $d(x_1, x_3) = |x_1 - x_3|$

$$= |x_1 - x_2 + x_2 - x_3|$$
$$\leq |x_1 - x_2| + |x_2 - x_3|$$
$$= d(x_1, x_2) + d(x_2, x_3)$$

Thus $(R; d)$ is a metric space.

Example

Let $X = R^2$ be a set of all ordered pairs (x, y); $x, y \in R$. Let $P_1(x_1, y_1), P_2(x_2, y_2) \in R^2$. Show that the non-negative real valued function "d" defined by $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$ is a metric on R^2

Solution

Here function $d: R^2 \times R^2 \to R$ is defined as

$$d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$$

(M$_1$) Since $|x_1 - x_2| + |y_1 - y_2| \geq 0$

$$\therefore d(P_1, P_2) \geq 0$$

(M$_2$) Let $d(P_1, P_2) = 0 \Rightarrow |x_1 - x_2| + |y_1 - y_2| = 0$

$$\Rightarrow |x_1 - x_2| = 0, \quad |y_1 - y_2| = 0$$
$$\Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0$$
$$\Rightarrow x_1 = x_2, \quad y_1 = y_2$$
$$\Rightarrow (x_1, y_1) = (x_2, y_2)$$
$$\Rightarrow P_1 = P_2$$

Let $P_1 = P_2 \Rightarrow (x_1, y_1) = (x_2, y_2)$

$$\Rightarrow x_1 = x_2, \quad y_1 = y_2$$
$$\Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0$$
$$\Rightarrow |x_1 - x_2| = 0, \quad |y_1 - y_2| = 0$$
$$\Rightarrow |x_1 - x_2| + |y_1 - y_2| = 0$$
$$\Rightarrow d(P_1, P_2) = 0$$

Thus $d(P_1, P_2) = 0 \Leftrightarrow P_1 = P_2$

(M$_3$) Since $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$

$$= |-(x_2 - x_1)| + |-(y_2 - y_1)|$$
$$= |x_2 - x_1| + |y_2 - y_1|$$
$$= d(P_2, P_1)$$

(M$_4$) Since $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$

$$d(P_2, P_3) = |x_2 - x_3| + |y_2 - y_3|$$
$$d(P_1, P_3) = |x_1 - x_3| + |y_1 - y_3|$$

Since $d(P_1, P_3) = |x_1 - x_3| + |y_1 - y_3|$

$$= |x_1 - x_2 + x_2 - x_3| + |y_1 - y_2 + y_2 - y_3|$$
$$\leq |x_1 - x_2| + |x_2 - x_3| + |y_1 - y_2| + |y_2 - y_3|$$
$$= |x_1 - x_2| + |y_1 - y_2| + |x_2 - x_3| + |y_2 - y_3|$$
$$= d(P_1, P_2) + d(P_2, P_3)$$

Hence "d" is metric on R^2.

Example

Let $X = R^2$ be a set of all ordered pairs(x, y); $x, y \in R$. Let $P_1(x_1, y_1), P_2(x_2, y_2) \in R^2$. Show that the non-negative real valued function"d" defined by $d(P_1, P_2) = max\,(|x_1 - x_2|, |y_1 - y_2|)$ is a metric on R^2.

Solution

Here function $d: R^2 \times R^2 \to R$ is defined as

$$d(P_1, P_2) = max\,(|x_1 - x_2| + |y_1 - y_2|)$$

(M$_1$) Since $max\,(|x_1 - x_2| + |y_1 - y_2|) \geq 0$

$(\because |x_1 - x_2| \geq 0 \;\&\; |y_1 - y_2| \geq 0)$

$\therefore d(P_1, P_2) \geq 0$

(M$_2$) Let $d(P_1, P_2) = 0 \Rightarrow max\,(|x_1 - x_2|, |y_1 - y_2|) = 0$

$\Rightarrow |x_1 - x_2| = 0, \quad |y_1 - y_2| = 0$

$\Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0$

$\Rightarrow x_1 = x_2, \quad y_1 = y_2$

$\Rightarrow (x_1, y_1) = (x_2, y_2)$

$\Rightarrow P_1 = P_2$

Let $P_1 = P_2 \Rightarrow (x_1, y_1) = (x_2, y_2)$

$\Rightarrow x_1 = x_2, \quad y_1 = y_2$

$\Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0$

$\Rightarrow |x_1 - x_2| = 0, \quad |y_1 - y_2| = 0$

$\Rightarrow max\,(|x_1 - x_2|, |y_1 - y_2|) = 0$

$\Rightarrow d(P_1, P_2) = 0$

Thus $d(P_1, P_2) = 0 \Leftrightarrow P_1 = P_2$

(M$_3$) Since $d(P_1, P_2) = max\,(|x_1 - x_2|, |y_1 - y_2|)$

$$= max(|-(x_2 - x_1)|, |-(y_2 - y_1)|)$$
$$= max(|x_2 - x_1|, |y_2 - y_1|)$$
$$= d(P_2, P_1)$$

(M$_4$) Since $d(P_1,P_2) = max\ (|x_1 - x_2|, |y_1 - y_2|) = |x_1 - x_2|$ (Say)

$d(P_2,P_3) = max\ (|x_2 - x_3|, |y_2 - y_3|) = |x_2 - x_3|$ (Say)

$d(P_1,P_3) = max\ (|x_1 - x_3|, |y_1 - y_3|) = |x_1 - x_3|$ (Say)

Now $d(P_1,P_3) = |x_1 - x_3|$

$= |x_1 - x_2 + x_2 - x_3|$

$\leq |x_1 - x_2| + |x_2 - x_3|$

$= d(P_1,P_2) + d(P_2,P_3)$

(We can get the same results in the remaining cases.)

Hence "d" is metric on R^2.

Example

Let $X = R^2$ be a set of all ordered pairs(x,y); $x,y \in R$. Let $P_1(x_1,y_1), P_2(x_2,y_2) \in R^2$. Show that the non-negative real valued function "d" defined by $d(P_1,P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}}$ is a metric on R^2.

Solution

Here function $d: R^2 \times R^2 \rightarrow R$ is defined as

$$d(P_1,P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}}$$

(M$_1$) Since $[(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}} \geq 0$

$\therefore \quad d(P_1,P_2) \geq 0$

(M$_2$) Let $d(P_1,P_2) = 0 \Rightarrow [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}} = 0$

$\Rightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 = 0$

$\Rightarrow (x_1 - x_2)^2 = 0, \quad (y_1 - y_2)^2 = 0$

$\Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0$

$\Rightarrow x_1 = x_2, \quad y_1 = y_2$

$\Rightarrow (x_1,y_1) = (x_2,y_2)$

$\Rightarrow P_1 = P_2$

Let $P_1 = P_2 \Rightarrow (x_1,y_1) = (x_2,y_2)$

$\Rightarrow x_1 = x_2, \quad y_1 = y_2$

$\Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0$

$\Rightarrow (x_1 - x_2)^2 = 0, \ (y_1 - y_2)^2 = 0$

$\Rightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 = 0$

$\Rightarrow [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}} = 0$

$\Rightarrow d(P_1,P_2) = 0$

(M$_3$) Since $d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}}$

$$= [\{-(x_2 - x_1)\}^2 + \{-(y_2 - y_1)\}^2]^{\frac{1}{2}}$$

$$= [(x_2 - x_1)^2 + (y_2 - y_1)^2]^{\frac{1}{2}}$$

$$= d(P_2, P_1)$$

(M$_4$) Let $P_1(x_1, y_1), P_2(x_2, y_2), P_3(x_3, y_3) \in R^2$ then P_1, P_2, P_3 are collinear or non-collinear.

If P_1, P_2, P_3 are collinear, then

$$d(P_1, P_2) + d(P_2, P_3) = d(P_1, P_3) \text{ -------- ----- } (1)$$

If P_1, P_2, P_3 are non-collinear, then they form a triangle and we know that, we know that sum of length of two sides of a triangle is always greater than the third side.

$$\therefore \quad d(P_1, P_2) + d(P_2, P_3) > d(P_1, P_3) \text{ -------- ----- } (2)$$

From (1) & (2) we get,

$$d(P_1, P_2) + d(P_2, P_3) \geq d(P_1, P_3)$$

Hence "d" is metric on R^2.

Example

Let $X = R^3$ be a set of all ordered pairs(x, y); $x, y \in R$. Let $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2) \in R^3$. Show that the non-negative real valued function "d" defined by $d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}}$ is a metric on R^3.

$+(z_1 - z_2)^2$

Solution

Here function $d: R^3 \times R^3 \to R$ is defined as

$$d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{\frac{1}{2}}$$

(M$_1$) Since $[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{\frac{1}{2}} \geq 0$

$$\therefore \quad d(P_1, P_2) \geq 0$$

(M$_2$) Let $d(P_1, P_2) = 0 \Rightarrow [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{\frac{1}{2}} = 0$

$$\Rightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = 0$$

$$\Rightarrow (x_1 - x_2)^2 = 0, (y_1 - y_2)^2 = 0, (z_1 - z_2)^2 = 0$$

$$\Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0, \quad z_1 - z_2 = 0$$

$$\Rightarrow x_1 = x_2, \quad y_1 = y_2 \quad z_1 = z_2$$

$$\Rightarrow (x_1, y_1, z_1) = (x_2, y_2, z_2)$$

$$\Rightarrow P_1 = P_2$$

Let $P_1 = P_2 \Rightarrow (x_1, y_1, z_1) = (x_2, y_2, z_2)$

$\Rightarrow \quad x_1 = x_2, \quad y_1 = y_2, \quad z_1 = z_2$

$\Rightarrow \quad x_1 - x_2 = 0, \quad y_1 - y_2 = 0, \quad z_1 - z_2 = 0$

$\Rightarrow \quad (x_1 - x_2)^2 = 0, (y_1 - y_2)^2 = 0, (z_1 - z_2)^2 = 0$

$\Rightarrow \quad (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = 0$

$\Rightarrow \quad [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{\frac{1}{2}} = 0$

$\Rightarrow \quad d(P_1, P_2) = 0$

(M_3) Since $d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{\frac{1}{2}}$

$= [\{-(x_2 - x_1)\}^2 + \{-(y_2 - y_1)\}^2 + \{-(z_2 - z_1)\}^2]^{\frac{1}{2}}$

$= [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{\frac{1}{2}}$

$= d(P_2, P_1)$

(M_4) Let $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2), P_3(x_3, y_3, z_3) \in R^3$ then P_1, P_2, P_3 are collinear or non-collinear.

If P_1, P_2, P_3 are collinear, then

$d(P_1, P_2) + d(P_2, P_3) = d(P_1, P_3)$ -------------- (1)

If P_1, P_2, P_3 are non-collinear, then they form a triangle and we know that, we know that sum of length of two sides of a triangle is always greater than the third side.

$\therefore \quad d(P_1, P_2) + d(P_2, P_3) > d(P_1, P_3)$ -------------- (2)

From (1) & (2) we get,

$d(P_1, P_2) + d(P_2, P_3) \geq d(P_1, P_3)$

Hence "d" is metric on R^3.

Example

Show that every non-empty set can be given a metric and hence can be converted into metric space.

Solution

Let X be any non-empty set.

Let $d_\circ : X \times X \to R$ be defined by

$$d_\circ(x_1, x_2) = \begin{cases} 0 & if\ x_1 = x_2 \\ 1 & if\ x_1 \neq x_2 \end{cases}$$

We shall prove that $d_\circ$ is a metric on X.

(M_1) Here $d_\circ(x_1, x_2) \geq 0 \quad (\because d_\circ(x_1, x_2) = 0$ or $d_\circ(x_1, x_2) = 1)$

(M_2) Let $d_\circ(x_1, x_2) = 0 \Rightarrow x_1 = x_2$ (By definition)

Let $x_1 = x_2 \Rightarrow d_\circ(x_1, x_2) = 0$ (By definition)

Thus $d_\circ(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$

(M_3) (i) Let $d_\circ(x_1,x_2)=0 \Rightarrow x_1=x_2$ (By definition)

$\Rightarrow x_2=x_1$

$\Rightarrow d_\circ(x_2,x_1)=0$

(ii) Let $d_\circ(x_1,x_2)=1 \Rightarrow x_1\neq x_2$ (By definition)

$\Rightarrow x_2\neq x_1$

$\Rightarrow d_\circ(x_2,x_1)=1$

Hence in both the cases $d_\circ(x_1,x_2)=d_\circ(x_2,x_1)$

(M_4) Let $x_1,x_2,x_3\in X$

(i) Let $x_1=x_2=x_3$ then $d_\circ(x_1,x_2)=0$

& $d_\circ(x_2,x_3)=0$

also $d_\circ(x_1,x_3)=0$

$\therefore\quad d(x_1,x_2)+d(x_2,x_3)=d(x_1,x_3)$

(ii) Let $x_1\neq x_2\neq x_3$ then $d_\circ(x_1,x_2)=1$

& $d_\circ(x_2,x_3)=1$

also $d_\circ(x_1,x_3)=1$

$\therefore\quad \underbrace{d(x_1,x_2)+d(x_2,x_3)}_{1+1}>d(x_1,x_3)$

Similar type of verification in all remaining cases leads us to the conclusion that $\quad d(x_1,x_2)+d(x_2,x_3)\geq d(x_1,x_3)\quad \forall\, x_1,x_2,x_3\in X$

Hence $(X,d_\circ)$ is a metric space.

Note

Let X be any non-empty set. Let $d_\circ : X\times X\to R$ be defined by

$$d_\circ(x_1,x_2)=\begin{cases}0 & if\ x_1=x_2\\ 1 & if\ x_1\neq x_2\end{cases}$$

Then $d_\circ$ is called discrete metric on X.

Question

Let C be the set of all complex numbers and let $d: C\times C\to R$ be defined by $d(z_1,z_2)=|z_1-z_2|$ d is a metric on C

Solution

Here function $d: C\times C\to R$ is defined as

$d(z_1,z_2)=|z_1-z_2|$

(M_1) Since $|z_1-z_2|\geq 0$

$\therefore\quad d(z_1,z_2)\geq 0$

(M_2) Let $d(z_1,z_2)=0 \Rightarrow |z_1-z_2|=0$

$\Rightarrow z_1-z_2=0$

$\Rightarrow z_1=z_2$

$$\text{Let } z_1 = z_2 \Rightarrow z_1 - z_2 = 0$$
$$\Rightarrow |z_1 - z_2| = 0$$
$$\Rightarrow d(z_1, z_2) = 0$$

Thus $d(z_1, z_2) = 0 \Leftrightarrow z_1 = z_2$

(M$_3$) Since $d(z_1, z_2) = |z_1 - z_2|$

$$= |-(z_2 - z_1)|$$
$$= |z_2 - z_1|$$
$$= d(z_2, z_1)$$

(M$_4$) Since $d(z_1, z_2) = |z_1 - z_2|$

$$d(z_2, z_3) = |z_2 - z_3|$$
$$d(z_1, z_3) = |z_1 - z_3|$$

Now $d(z_1, z_3) = |z_1 - z_3|$

$$= |z_1 - z_2 + z_2 - z_3|$$
$$\leq |z_1 - z_2| + |z_2 - z_3|$$
$$= d(z_1, z_2) + d(z_2, z_3)$$

Thus (C, d) is a metric space.

Question

Let d be a metric on X and let $d': X \times X \to R$ be given by $d'(x_1, x_2) = min\,(1, d(x_1, x_2))$. Is d' a metric on X?

Solution

Here function $d': X \times X \to R$ be given by

$$d'(x_1, x_2) = min\,(1, d(x_1, x_2))$$

(M$_1$) Since $min\,(1, d(x_1, x_2)) \geq 0$

$\therefore \quad d'(x_1, x_2) \geq 0$

(M$_2$) Let $d'(x_1, x_2) = 0 \Rightarrow min\,(1, d(x_1, x_2)) = 0$

$\Rightarrow d(x_1, x_2) = 0 \qquad \because 1 \neq 0$

$\Rightarrow x_1 = x_2 \qquad \because d$ is metric on X.

Let $x_1 = x_2 \Rightarrow d(x_1, x_2) = 0 \qquad \because d$ is metric on X.

$\Rightarrow min\,(1, d(x_1, x_2)) = 0$

$\Rightarrow d'(x_1, x_2) = 0$

Thus $d'(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$

(M$_3$) Since $d'(x_1, x_2) = min\,(1, d(x_1, x_2))$

$= min\,(1, d(x_2, x_1)) \qquad \because d$ is metric on X.

$= d'(x_2, x_1)$

(M$_4$) Since $d'(x_1, x_2) = min\,(1, d(x_1, x_2)) = d(x_1, x_2)$ (Say)

$d'(x_2, x_3) = min\,(1, d(x_2, x_3)) = d(x_2, x_3)$ (Say)

$d'(x_1, x_3) = min\,(1, d(x_1, x_3)) = d(x_1, x_3)$ (Say)

Since d is a metric on X.

$\therefore \quad d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3)$

$\Rightarrow \quad d'(x_1, x_2) + d'(x_2, x_3) \geq d'(x_1, x_3)$

We get the same result in the remaining cases.

$\therefore \quad d'$ is a metric on X.

Question

Let (X_1, d_1) and (X_2, d_2) be two metric spaces

Define $d'[(x_1, x_2), (y_1, y_2)] = \sum_{i=1}^{2} d_i(x_i, y_i)$. Is d' a metric on $X_1 \times X_2$.

Solution

Here function $d' : X_1 \times X_2 \to R$ is defined as

$$d'[(x_1, x_2), (y_1, y_2)] = \sum_{i=1}^{2} d_i(x_i, y_i)$$
$$= d_1(x_1, y_1) + d_2(x_2, y_2)$$

(M$_1$) Since $d_1(x_1, y_1) + d_2(x_2, y_2) \geq 0$

$\because \quad d_1(x_1, y_1) \geq 0, \quad d_2(x_2, y_2) \geq 0$

$\because \quad d_1, d_2$ are metrics on X_1 and X_2 respectively.

$\therefore \quad d'\big((x_1, x_2), (y_1, y_2)\big) \geq 0$

(M$_2$) Let $d'\big((x_1, x_2), (y_1, y_2)\big) = 0 \quad \Rightarrow \quad d_1(x_1, y_1) + d_2(x_2, y_2) = 0$

$\Rightarrow \quad d_1(x_1, y_1) = 0, \quad d_2(x_2, y_2) = 0$

$\Rightarrow \quad x_1 = y_1, \qquad x_2 = y_2$

$\because \quad d_1, d_2$ are metrics on $X_1 \times X_2$

$\Rightarrow \quad (x_1, x_2) = (x_2, y_2)$

Let $(x_1, x_2) = (y_1, y_2) \quad \Rightarrow \quad x_1 = y_1, \qquad x_2 = y_2$

$\Rightarrow \quad d_1(x_1, y_1) = 0 \quad d_2(x_2, y_2) = 0$

($\because$ d_1, d_2 are metrics on X_1 & X_2 respectively)

$\Rightarrow \quad d_1(x_1, y_1) + d_2(x_2, y_2) = 0$

$\Rightarrow \quad d'\big((x_1, x_2), (y_1, y_2)\big) = 0$

Thus $d'\big((x_1, x_2), (y_1, y_2)\big) = 0 \iff (x_1, x_2) = (y_1, y_2)$

(M$_3$) Since $d'\big((x_1, x_2), (y_1, y_2)\big) = d_1(x_1, y_1) + d_2(x_2, y_2)$

$$= d_1(y_1, x_1) + d_2(y_2, x_2)$$

$$= d'\big((y_1, y_2), (x_1, x_2)\big)$$

(M$_4$) Since $d'\big((x_1, x_2), (y_1, y_2)\big) = d_1(x_1, y_1) + d_2(x_2, y_2)$

$d'\big((y_1, y_2), (z_1, z_2)\big) = d_1(y_1, z_1) + d_2(y_2, z_2)$

$d'\big((x_1, x_2), (z_1, z_2)\big) = d_1(x_1, z_1) + d_2(x_2, z_2)$

Now $d'\big((x_1, x_2), (y_1, y_2)\big) + d'\big((y_1, y_2), (z_1, z_2)\big) = d_1(x_1, y_1) + d_2(x_2, y_2)$

$$+ d_1(y_1, z_1) + d_2(y_2, z_2)$$
$$= d_1(x_1, y_1) + d_1(y_1, z_1)$$
$$+ d_2(x_2, y_2) + d_2(y_2, z_2)$$
$$\geq d_1(x_1, z_1) + d_2(x_2, z_2)$$

$$\left(\begin{array}{l} \because d_1, d_2 \text{ are metrics on } X_1 \text{ \& } X_2 \text{ resp.} \\ \therefore d_1(x_1, y_1) + d_1(y_1, z_1) \geq d_1(x_1, z_1) \\ \text{\& } d_2(x_2, y_2) + d_1(y_2, z_2) \geq d_2(x_2, z_2) \end{array}\right)$$

$$= d'\big((x_1, x_2), (z_1, z_2)\big)$$

$\therefore \quad d'$ is a metric on $X_1 \times X_2$.

Question

Let (X_1, d_1) and (X_2, d_2) be two metric spaces.

Let $d''[(x_1, x_2), (y_1, y_2)] = max\left(d_1(x_1, y_1), d_2(x_2, y_2)\right)$.

Is d'' a metric on $X_1 \times X_2$.

Solution

Here function $d'': X_1 \times X_2 \to R$ is defined as

$$d''[(x_1, x_2), (y_1, y_2)] = max\left(d_1(x_1, y_1), d_2(x_2, y_2)\right)$$

(M$_1$) Since $max(d_1(x_1, y_1) + d_2(x_2, y_2) \geq 0)$

$\because \quad d_1(x_1, y_1) \geq 0, \quad d_2(x_2, y_2) \geq 0$

$\because$ d_1, d_2 are metrics on X_1 and X_2 respectively

$\therefore \quad d''\left((x_1, x_2), (y_1, y_2)\right) \geq 0$.

(M$_2$) Let $d''\left((x_1, x_2), (y_1, y_2)\right) = 0 \quad \Rightarrow \quad max\left(d_1(x_1, y_1), d_2(x_2, y_2)\right) = 0$

$\Rightarrow \quad d_1(x_1, y_1) = 0, \quad d_2(x_2, y_2) = 0$

$\Rightarrow \quad x_1 = y_1, \qquad x_2 = y_2$

($\because$ d_1, d_2 are metrics on X_1, X_2 respectively)

$\Rightarrow \quad (x_1, x_2) = (y_1, y_2)$

Let $(x_1, x_2) = (y_1, y_2) \quad \Rightarrow \quad x_1 = y_1, \qquad x_2 = y_2$

$\Rightarrow \quad d_1(x_1, y_1) = 0, \quad d_2(x_2, y_2) = 0$

($\because$ d_1, d_2 are metrics on X_1, X_2 respectively)

$\Rightarrow \quad max\left(d_1(x_1, y_1), d_2(x_2, y_2)\right) = 0$

$\Rightarrow \quad d''\left((x_1, x_2), (y_1, y_2)\right) = 0$

$d''\left((x_1, x_2), (y_1, y_2)\right) = 0 \quad \Leftrightarrow \quad (x_1, x_2) = (y_1, y_2)$.

(M$_3$) Since $d''[(x_1, x_2), (y_1, y_2)] = max\left(d_1(x_1, y_1), d_2(x_2, y_2)\right)$

$= max\left(d_1(y_1, x_1), d_2(y_2, x_2)\right)$

($\because$ d_1, d_2 are metrics on X_1, X_2 respectively)

$= d''\left((y_1, y_2), (x_1, x_2)\right)$

(M$_4$) Let. $d''[(x_1, x_2), (y_1, y_2)] = max\left(d_1(x_1, y_1), d_2(x_2, y_2)\right) = d_1(x_1, y_1)$ (Say)

$d''[(y_1, y_2), (z_1, z_2)] = max\left(d_1(y_1, z_1), d_2(y_2, z_2)\right) = d_1(y_1, z_1)$ (Say)

$d''[(x_1, x_2), (z_1, z_2)] = max\left(d_1(x_1, z_1), d_2(x_2, z_2)\right) = d_1(x_1, z_1)$ (Say)

Since d_1 is a metric on X_1.

$\therefore \quad d_1(x_1, y_1) + d_1(y_1, z_1) \geq d_1(x_1, z_1)$

$\Rightarrow d''[(x_1, x_2), (y_1, y_2)] + d''[(y_1, y_2), (z_1, z_2)] \geq d''[(x_1, x_2), (z_1, z_2)]$

(We get the same result in the remaining cases.)

$\therefore \quad d''$ is a metric on $X_1 \times X_2$.

Question

Let (X, d) be a metric space and let $d': X \times X \rightarrow R$ be given by $d'(x_1, x_2) = \frac{d(x_1,x_2)}{1 + d(x_1,x_2)}$. Prove that d' is metric on X.

Solution

Here function $d': X \times X \rightarrow R$ be defined by

$$d'(x_1, x_2) = \frac{d(x_1,x_2)}{1+d(x_1,x_2)}$$

(M$_1$) Since $\frac{d(x_1,x_2)}{1+d(x_1,x_2)} \geq 0$

$\because$ $d(x_1, x_2) \geq 0$

$\because$ d is a metric on X.

$\therefore$ $d'(x_1, x_2) \geq 0$

(M$_2$) Let $d'(x_1, x_2) = 0 \Rightarrow \frac{d(x_1,x_2)}{1+d(x_1,x_2)} = 0$

$\Rightarrow d(x_1, x_2) = 0$

$\Rightarrow x_1 = x_2$ $\quad (\because d$ is a metric on $X.)$

Let $x_1 = x_2 \Rightarrow d(x_1, x_2) = 0$ $\quad (\because d$ is a metric on $X.)$

$\Rightarrow \frac{d(x_1,x_2)}{1+d(x_1,x_2)} = 0$

$\Rightarrow d'(x_1, x_2) = 0$

Thus $d'(x_1, x_2) = 0 \quad \Leftrightarrow \quad x_1 = x_2$

(M$_3$) Since $d'(x_1, x_2) = \frac{d(x_1,x_2)}{1+d(x_1,x_2)}$ $\quad \therefore d$ is metric on x

$= \frac{d(x_2,x_1)}{1+d(x_2,x_1)}$

$= d'(x_2, x_1)$

(M$_4$) Since $d'(x_1, x_2) = \frac{d(x_1,x_2)}{1+d(x_1,x_2)}$

$d'(x_2, x_3) = \frac{d(x_2,x_3)}{1+d(x_2,x_3)}$

$d'(x_1, x_3) = \frac{d(x_1,x_3)}{1+d(x_1,x_3)}$

Now $d'(x_1, x_2) + d'(x_2, x_3) = \frac{d(x_1,x_2)}{1 + d(x_1,x_2)} + \frac{d(x_2,x_3)}{1 + d(x_2,x_3)}$

$\geq \frac{d(x_1,x_2)}{1 + d(x_1,x_2) + d(x_2,x_3)} + \frac{d(x_2,x_3)}{1 + d(x_1,x_2) + d(x_2,x_3)}$

$= \frac{d(x_1,x_2) + d(x_2,x_3)}{1 + d(x_1,x_2) + d(x_2,x_3)}$

$\therefore$ $d'(x_1, x_2) + d'(x_2, x_3) \geq \frac{d(x_1,x_3)}{1 + d(x_1,x_3)}$ $\left(\begin{array}{l} \because d(x_1,x_2) + d(x_2,x_3) \geq d(x_1,x_3) \\ \because d \text{ is a metric on } X. \end{array} \right)$

$= d'(x_1, x_3)$

$\therefore$ d' is a metric on X.

Question

Let $X = R$ and $d(x_1, x_2) = |x_1| + |x_2|$. Show that d is not a metric on R.

Solution

Let $d(x_1, x_2) = 0 \Rightarrow |x_1| + |x_2| = 0$

$\Rightarrow |x_1| = 0, \quad |x_2| = 0$

$\Rightarrow x_1 = 0, \quad x_2 = 0$

$\Rightarrow x_1 = x_2$

Let $x_1 = x_2 \Rightarrow |x_1| = |x_2|$

$\Rightarrow |x_1| + |x_2| = |x_2| + |x_2|$ (Adding $|x_2|$ both sides)

$\Rightarrow d(x_1, x_2) = 2|x_2|$

$\Rightarrow d(x_1, x_2) = 0$ if $|x_2| = 0$

i.e. $d(x_1, x_2)$ is not always zero.

$\because$ d is not a metric on X.

Question

Let $X = R$ and $d(x_1, x_2) = max\,(|x_1|, |x_2|)$. Show that d is not a metric on R.

Solution

Let $d(x_1, x_2) = 0 \Rightarrow max\,(|x_1|, |x_2|) = 0$

$\Rightarrow |x_1| = 0, \quad |x_2| = 0$

$\Rightarrow x_1 = 0, \quad x_2 = 0$

$\Rightarrow x_1 = x_2$

Let $x_1 = x_2 \Rightarrow |x_1| = |x_2|$

$\Rightarrow max\,(|x_1|, |x_2|) = |x_2|$

$\Rightarrow d(x_1, x_2) = 0$ if $|x_2| = 0$

i.e. $d(x_1, x_2)$ is not always zero.

Thus d is not a metric on X.

Question

Let (X, d) be a metric space and let $d'': X \times X \to R$ be given by $d''(x_1, x_2) = \frac{1 - d(x_1,x_2)}{1 + d(x_1,x_2)}$. Prove that d'' is not a metric on X.

Solution

Let $d''(x_1, x_2) = 0 \Rightarrow \frac{1 - d(x_1,x_2)}{1 + d(x_1,x_2)} = 0$

$\Rightarrow 1 - d(x_1, x_2) = 0$

$\Rightarrow d(x_1, x_2) = 1$

$\Rightarrow x_1 \neq x_2$

$\because$ d is a metric on X and $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$

Thus $d''(x_1, x_2) = 0 \not\Rightarrow x_1 = x_2$

Thus d'' is not a metric on X.

S

OPEN SPHERE

Open sphere

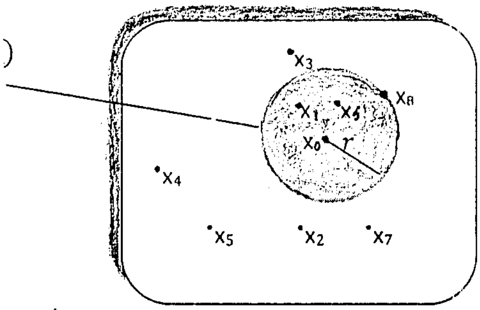
Let (X, d) be a metric space. Let $x_\circ \in X$ and $r > 0$, then open sphere with centre at $x_\circ$ and radius equal to r is denoted by $S_r(x_\circ)$ and is defined as $S_r(x_\circ) = \{\, x | x \in X,\ d(x, x_\circ) < r \,\}$

Note

(i) Let $X = \{\, x_\circ, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \,\}$ and $r > 0$

$S_r(x_\circ) = ?$

$S_r(x_\circ)$



X

Then by definition of open sphere $S_r(x_\circ) = \{\, x_\circ, x_1, x_6 \,\}$

(ii) $S_r(x_\circ) \subseteq X$

(iii) $S_r(x_\circ) \neq \phi$

(iv) Here we shall study the open spheres of the following shapes.

(a) Open interval (b) Open disc (c) Open ball

The shape of an open sphere depends upon the metric space (X, d).

Example

Let R be the metric space. Let $x_0 = 1$, $r = \frac{1}{2}$. Find $S_{\frac{1}{2}}(1)$.

Solution

Here metric space is (R, d), where metric $d: R \times R \to R$ is defined as $d(x_1, x_2) = |x_1 - x_2|$

We know that

$S_r(x_\circ) = \{\, x | x \in X,\ d(x, x_\circ) < r \,\}$

Put $X = R,\quad x_\circ = 1,\quad r = \frac{1}{2}$

$$\therefore\ S_{\frac{1}{2}}(1) = \left\{ x | x \in R,\ d(x, 1) < \tfrac{1}{2} \right\}$$
$$= \left\{ x | x \in R,\ |x - 1| < \tfrac{1}{2} \right\}$$
$$= \left\{ x | x \in R,\ x - 1 < \tfrac{1}{2},\ x - 1 > -\tfrac{1}{2} \right\}$$
$$= \left\{ x | x \in R,\ x < 1 + \tfrac{1}{2},\ x > 1 - \tfrac{1}{2} \right\}$$
$$= \left\{ x | x \in R,\ \tfrac{1}{2} < x < \tfrac{3}{2} \right\}$$
$$= \left] \tfrac{1}{2}, \tfrac{3}{2} \right[$$

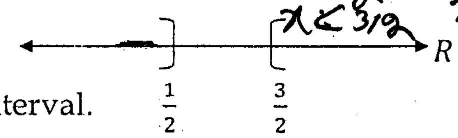
$|x-1| < 1/2$

$\pm(x-1) < \frac{1}{2}$

$x-1 < \frac{1}{2}$ or $-(x-1) < \frac{1}{2}$

$x < 1 + 1/2$ or $x - 1 > -1/2$

$x < 3/2$ $\quad x > 1/2$



Open sphere in this case is an open interval.

K·S

Note

An open sphere in a usual metric space R is always an "open interval".

Example

Let the metric space be R^2 and let $P_\circ = (a, b)$ and $r = 1$. Find $S_r(P_\circ)$.

Solution

Here metric space is (R, d), where metric $d: R \times R \to R$ is defined as $d[(x_1, y_1), (x_2, y_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

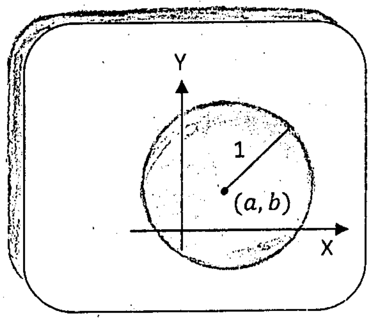
We know that

$$S_r(P_\circ) = \{ P | P \in X,\ d(P, P_\circ) < r \}$$ K.S

Put $X = R^2, \quad P_\circ = (a, b), \quad P = (x, y), \quad r = 1$

$$\therefore S_1(a, b) = \{ (x, y) | (x, y) \in R^2,\ d((x, y), (a, b)) < 1 \}$$
$$= \left\{ (x, y) | (x, y) \in R^2,\ \sqrt{(x - a)^2 + (y - b)^2} < 1 \right\}$$
$$= \{ (x, y) | (x, y) \in R^2,\ (x - a)^2 + (y - b)^2 < 1 \}$$

This is an open disc with centre at (a, b) and radius 1.



R^2

Note

An open sphere in a usual metric space R^2 is always an "open disc".

Example

Let the metric space be R^2 and d_1 be the metric on R^2 defined by
$$d_1(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|.$$
Let $P_\circ = (0, 0)$ and $r = \frac{1}{\sqrt{2}}$. Find $S_r(x_\circ)$.

Solution

Here metric space is (R^2, d_1), where metric $d_1: R^2 \times R^2 \to R$ is defined as $d_1[(x_1, y_1), (x_2, y_2)] = |x_1 - x_2| + |x_1 - x_2|$

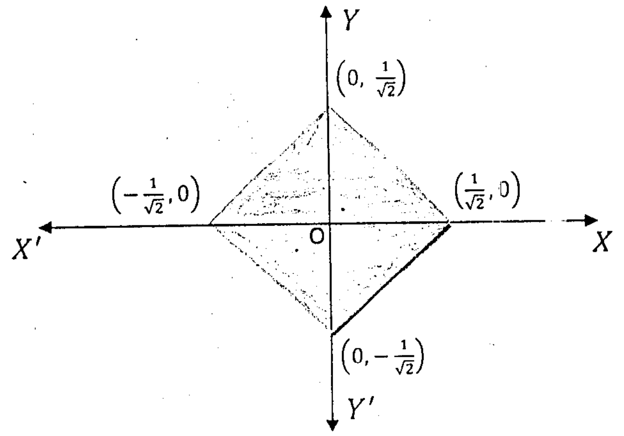
We know that

$$S_r(P_\circ) = \{ P | P \in X,\ d(P, P_\circ) < r \}$$ K.S

Put $X = R^2, \quad P_\circ = (0,0), \quad P = (x, y), \quad r = \frac{1}{\sqrt{2}}$

$$\therefore S_{\frac{1}{\sqrt{2}}}(0,0) = \left\{ (x, y) | (x, y) \in R^2,\ d_1((x, y), (0,0)) < \frac{1}{\sqrt{2}} \right\}$$
$$= \left\{ (x, y) | (x, y) \in R^2,\ |x - 0| + |y - 0| < \frac{1}{\sqrt{2}} \right\}$$
$$= \left\{ (x, y) | (x, y) \in R^2,\ |x| + |y| < \frac{1}{\sqrt{2}} \right\}$$ — End which is required open sphere.
$$= \left\{ (x, y) | (x, y) \in R^2,\ \pm x \pm y < \frac{1}{\sqrt{2}} \right\}$$
$$= \left\{ (x, y) | (x, y) \in R^2,\ \frac{x}{\pm\frac{1}{\sqrt{2}}} \pm \frac{y}{\pm\frac{1}{\sqrt{2}}} < 1 \right\}$$

This is an open square with x-intercepts $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$ and y-intercepts $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$



Example

Let $(X, d_\circ)$ be a discrete metric space. Let $x_\circ \in X$ and $r > 0$

Find $S_r(x_\circ)$, when (i) $r \leq 1$ (ii) $r > 1$

Solution

Here metric space is $(X, d_\circ)$, where $d_\circ : X \times X \to R$ is defined by

$$d_\circ(x_1, x_2) = \begin{cases} 0 & if\ x_1 = x_2 \\ 1 & if\ x_1 \neq x_2 \end{cases}$$

We know that

$$S_r(x_\circ) = \{ x | x \in X,\ d(x, x_\circ) < r \} \text{ ——————— } (1)$$

1, 0

When $r \leq 1$

If $x \neq x_\circ$ then from equation (1) we get $1 < r$ (False)

If $x = x_\circ$ then from equation (1) we get $0 < r$ (True)

Thus $S_r(x_\circ) = \{ x | x \in X,\ x = x_\circ \} = \{ x_\circ \}$

When $r > 1$

If $x \neq x_\circ$ then from equation (1) we get $1 < r$ (True)

If $x = x_\circ$ then from equation (1) we get $0 < r$ (True)

Thus $S_r(x_\circ) = \{ x | x \in X,\ x = x_\circ \text{ or } x \neq x_\circ \}$

$= \{ x | x \in X,\ x = x_\circ \} \cup \{ x | x \in X,\ x \neq x_\circ \}$

$= \{ x_\circ \} \cup X - \{ x_\circ \}$

$= X$

Note

From above example we conclude that

(i) An open sphere with radius less than or equal to 1 in a discrete metric space is always singleton.

(ii) An open sphere with radius greater than 1 in a discrete metric space is always the full space X.

A+

Question

Let C be the set of all complex numbers and let $d: C \times C \to R$ be defined by $d(z_1, z_2) = |z_1 - z_2|$. Find $S_r(x_\circ)$ when $x_\circ = 1,\ r = 0.01$

Solution

The given metric space is (C, d), where $d: C \times C \to R$ be defined by

$$d(z_1, z_2) = |z_1 - z_2|$$

Now $\quad S_r(x_\circ) = \{\, x | x \in X,\ d(x, x_\circ) < r \,\}$

Put $X = C, \quad x_\circ = 1 \qquad r = 0.01$

$\therefore\ S_{0.01}(1) = \{\, x | x \in C,\ d(x, 1) < 0.01 \}$

$= \{\, x | x \in C,\ |x - 1| < 0.01 \}$ ———— (1)

$|x-1| < 0.01$
$\pm(x-1) < 0.01$
$x-1 < 0.01;\ -(x-1) < 0.01$
$x < 1+0.01;\ x-1 > -0.01$
$x < 1.01;\ x > 0.99$
$]0.99, 1.01[$
S_0

Since $x \in C \quad \therefore\ x = a + ib$

$\Rightarrow \quad x - 1 = a + ib - 1$

$\Rightarrow \quad x - 1 = (a - 1) + ib$

$\Rightarrow \quad |x - 1| = \sqrt{(a - 1)^2 + b^2}$

$\therefore (1) \Rightarrow S_{0.01}(1) = \left\{ (a + ib) | (a + ib) \in C,\ \sqrt{(a - 1)^2 + b^2} < 0.01 \right\}$

$= \{(a + ib) | (a + ib) \in C, (a - 1)^2 + (b - 0)^2 < (.01)^2\}$

This is an open disc with centre at (1,0) and radius equal to 0.01.

Question

Let d be a metric on X and let $d': X \times X \to R$ be given by $d'(x_1, x_2) = min\left(1, d(x_1, x_2)\right)$. Describe $S_r(x_\circ)$.

Solution

Here given metric space is (X, d'), where $d': X \times X \to R$ be given by

$$d'(x_1, x_2) = min\left(1, d(x_1, x_2)\right)$$

Now $\qquad S_r(x_\circ) = \{\, x | x \in X,\ d'(x, x_\circ) < r \}$

$= \left\{\, x | x \in X,\ min\left(1, d(x_1, x_2)\right) < r \right\}$

This is the required open sphere.

Question

Let (X, d) be a metric space and let $d': X \times X \to R$ be given by $d'(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}$. Describe $S_r(x_\circ)$.

Solution

Here given metric space is (X, d'), where $d': X \times X \to R$ be given by

$$d'(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}$$

Now $\qquad S_r(x_\circ) = \{\, x | x \in X,\ d'(x, x_\circ) < r \,\}$

$= \left\{\, x | x \in X,\ \frac{d(x_1, x_2)}{1 + d(x_1, x_2)} < r \right\}$

This is the required open sphere.

A11

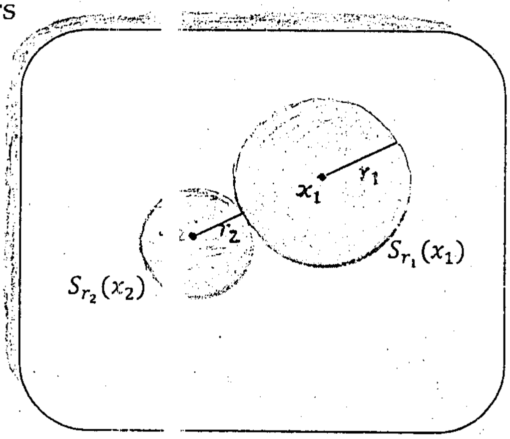
Theorem

Let x_1, x_2 be any two distinct points of a metric space X. Prove that there exist two open spheres $S_{r_1}(x_1)$ and $S_{r_2}(x_2)$ in X such that

$$S_{r_1}(x_1) \cap S_{r_2}(x_2) = \phi$$

Proof

Let $S_{r_1}(x_1)$ and $S_{r_2}(x_2)$ be two open spheres with centers x_1 and x_2 and radii r_1 and r_2 respectively.



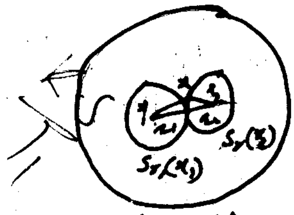
Let $d(x_1, x_2) = r_1 + r_2$

We are to prove that

$$S_{r_1}(x_1) \cap S_{r_2}(x_2) = \phi$$

We shall prove it by contradiction method.

Suppose $\quad S_{r_1}(x_1) \cap S_{r_2}(x_2) \neq \phi$



Let $\quad x \in S_{r_1}(x_1) \cap S_{r_2}(x_2)$

$\Rightarrow \quad x \in S_{r_1}(x_1) \quad$ and $\quad x \in S_{r_2}(x_2)$

$\Rightarrow \quad d(x, x_1) < r_1 \quad$ and $\quad d(x, x_2) < r_2 \quad$ ———— (1) (x, d)

Since $\quad r_1 + r_2 = d(x_1, x_2) \leq d(x_1, x) + d(x, x_2)$

$\because$ d is a metric on X.

$\Rightarrow \quad r_1 + r_2 \leq d(x_1, x) + d(x, x_2)$

$\Rightarrow \quad r_1 + r_2 < r_1 + r_2 \qquad$ [By (1)]

It is impossible.

Thus our supposition $S_{r_1}(x_1) \cap S_{r_2}(x_2) \neq \phi$ is wrong.

Hence $S_{r_1}(x_1) \cap S_{r_2}(x_2) = \phi$

Let x_1 and x_2 be any two pts of a metric space X
Let $S_r(x_1)$ and $S_r(x_2)$ be any open spheres in X
To prove $S_r(x_1) \cap S_r(x_2) = \phi$
Let $x \in S_r(x_1) \cap S_r(x_2)$
Thus by triangle inequality
Then $d(x_1, x_2) \leq d(x_1, x) + d(x, x_2)$
$r_1 + r_2 < r_1 + r_2$

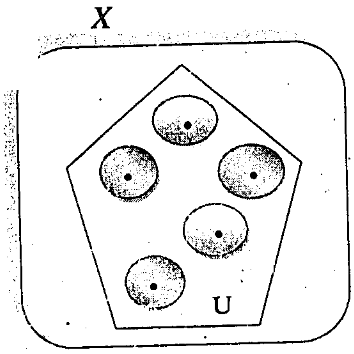
A subset U of a metric space (X, d) is called an open set if for every $x \in U$ there exists a real number $r > 0$ such that $x \in S_r(x) \subseteq U$

OPEN SET

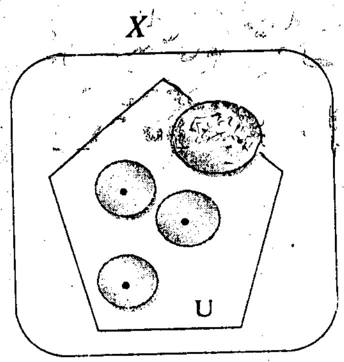
Open Set

Let (X, d) be a metric space. Let $U \subseteq X$. The U is called an open set, if for each $x \in U$, $\exists\, r > 0$, such that $S_r(x) \subseteq U$.

i.e. U is called an open set, if each point of U is the centre of some open sphere, which is contained in U.



U is an open set.



U is not an open set.

Example

Let R be a usual metric space (The ordinary real number line) and let U=$]0, 1[$, then show that U is open.

Solution

Here metric space is (R, d), where $d : R \times R \to R$ is given by

$$d(x_1, x_2) = |x_1 - x_2|$$

Let $x_\circ \in U$, Let $r > 0$

Then $S_r(x_\circ) = \{x | x \in R,\ d(x, x_\circ) < r\}$

$= \{x | x \in R,\ |x - x_\circ| < r\}$

$= \{x | x \in R,\ x - x_\circ < r,\ x - x_\circ > -r\}$

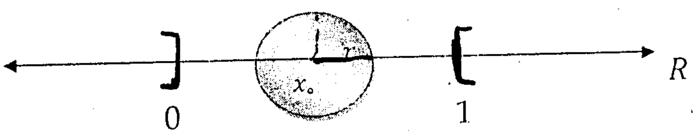
$= \{x | x \in R,\ x < x_\circ + r,\ x > x_\circ - r\}$

$= \{x | x \in R,\ x_\circ - r < x < x_\circ + r\}$

$=]x_\circ - r,\ x_\circ + r[$

We can find a value of r for which $S_r(x_\circ) =]x_\circ - r, x_\circ + r[\subseteq U =]0,1[$

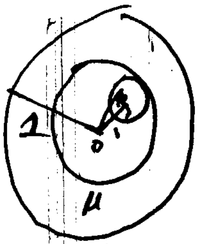
Thus U=$]0, 1[$ is an open set.



Note

In the above example if we take $x_\circ = 0.99$. Let $r = 0.001$

Then $S_{0.001}(0.99) =]0.99 - 0.001, 0.99 + 0.001[=]0.981, 0.991[\subseteq]0, 1[$



(X, d)

Example

Let R^2 be a usual metric space (The ordinary real plane)

Let U= $\{(x,y)|(x,y) \in R^2, x^2+y^2 < 1\}$. Show that U is an open set.

Solution

Let (X,d) be a metric space
Let $U = \{P \mid P \in R^2, x^2+y^2<1\}$
To prove U is open set
Let $P_0 \in U$
construct an open sphere centre at P_0 having radius equal to r.
Take $P \in S_r(P_0)$ such that $d(P,P_0)<r$
Join the pt O and P
Thus in this way we got a vector triangle
now from triangle inequality

Here metric space is (R^2, d), where $d: R^2 \times R^2 \to R$ is given by

$$d[(x_1,y_1),(x_2,y_2)] = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

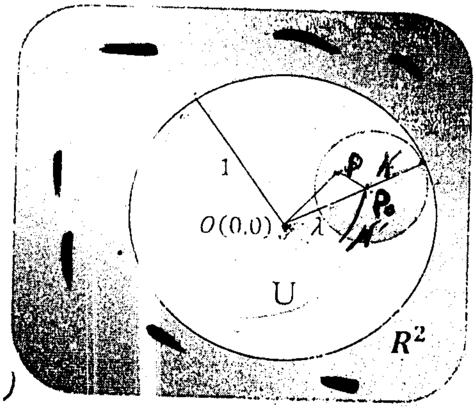
Let $P_\circ \in$ U.

Let $d(O, P_\circ) = \lambda$

Let $r = 1 - \lambda$, then $r > 0$

We shall prove that $S_r(P_\circ) \subseteq$ U

Let $P \in S_r(P_\circ) \Rightarrow d(P, P_\circ) < r$



Since (R^2, d) is a metric space,

$\therefore \quad d(P, P_\circ) + d(P_\circ, O) \geq d(P, O)$

$\Rightarrow \quad r + \lambda > d(P, O) \qquad \because r > d(P, P_0)$

$\Rightarrow \quad 1 - \lambda + \lambda > d(P, O) \qquad \because r = 1 - \lambda$

$\Rightarrow \quad d(P, O) < 1$

$\Rightarrow \quad P \in$ U

Since $P \in S_r(P_\circ) \Rightarrow P \in$ U

$\therefore \quad S_r(P_\circ) \subseteq$ U

Hence U is an open set.

$K = S$ $P_3 2$

Example

Let R be a usual metric space (The ordinary real number line) and let U= $\{x|x \in R, \ 0 \leq x < 1\}$, then show that U is not open.

Solution

Here metric space is (R, d), where $d: R \times R \to R$ is given by

$d(O,P) \le d(O,P_0) + d(P_0,P)$ $\quad d(x_1, x_2) = |x_1 - x_2|$

$d(O,P_0) = \lambda$ — ①

Here U= $\{x|x \in R, \ 0 \leq x < 1\}$

$= [0, 1[$

$d(P_0,P) < r$ — ②

$d(O,P) \le r + \lambda$

Let $x_\circ = 0 \in$ U, Let $r > 0$

$r + \lambda = 1$

Then $S_r(0) = \{x|x \in R, \ d(x,0) < r\}$

$d(O,P) \le 1$

$= \{x|x \in R, \ |x - 0| < r\}$

$= \{x|x \in R, \ |x| < r\}$

$d(O,P) < 1$

$= \{x|x \in R, \ x < r, \ x > -r\}$

$= \{x|x \in R, \ -r < x < r\}$

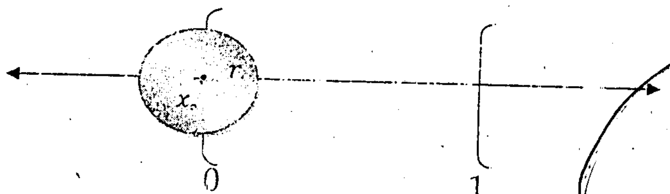
$=]-r, +r[$

$P \in U$

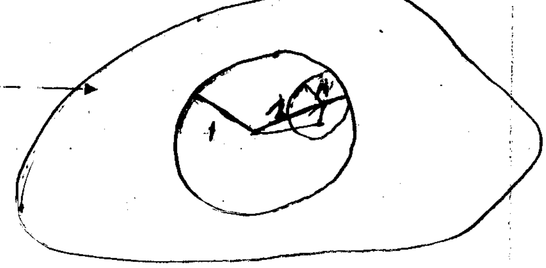
We can find a value of r for which $S_r(0) =]-r \ \ +r[\not\subseteq$ U=$[0,1[$

Thus U=$[0, 1[$ is not an open set.

$P \in S_r(P_0)$



$S_r(P_0) \subseteq U$



$= 1 - \lambda$

Theorem

Every non-empty subset of a discrete metric space is open.

Proof

Let $(X, d_\circ)$ be a discrete metric space.

Let $U \subseteq X$ such that $U \neq \phi$

We shall prove that U is an open set.

Let $x_\circ \in U$.

Let $0 < r < 1$

Then $S_r(x_\circ) = \{x | x \in X,\ d(x, x_\circ) < r\}$

$= \{x_\circ\}$.

$\therefore$ The open sphere in a discrete metric space, whose radius is less then 1, is always singleton. take centre

Since $S_r(x_\circ) = \{x_\circ\} \subseteq U$ Centre point

$\Rightarrow$ U is an open set.

Example

Let R be a usual metric space (The ordinary real number line) and let U= { 0 }, then show that U is not open.

Solution

Here metric space is (R, d), where $d: R \times R \to R$ is given by

$$d(x_1 . x_2) = |x_1 - x_2|$$

Here U= { 0 }

Let $x_\circ = 0 \in U$, Let $r > 0$

Then $S_r(0) = \{x | x \in R,\ d(x, 0) < r\}$

$= \{x | x \in R,\ |x - 0| < r\}$

$= \{x | x \in R,\ |x| < r\}$

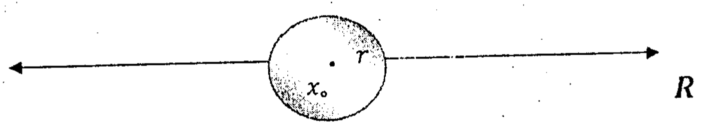
$= \{x | x \in R,\ x < r,\ x > -r\}$

$= \{x | x \in R,\ -r < x < r\}$

$=]-r, +r[$

We can find a value of r for which $S_r(0) =]-r, +r[\nsubseteq U = \{0\}$

Thus U= { 0 } is not an open set.



Theorem

Let (X, d) be a metric space, then

(i) Union of any collection $\{ U_\alpha : \alpha \in I \}$ of open sets is open.

(ii) Intersection of finite number of open sets is open.

(iii) The Whole space X and the empty set ϕ are both open.

Proof

(i) Let $\{ U_\alpha : \alpha \in I \}$ be any collection of open sets in (X, d).

We are to prove that, $\bigcup_{\alpha \in I} U_\alpha$ is an open set.

Let $x \in \bigcup_{\alpha \in I} U_\alpha$

Then $x \in U_\alpha$ for some $\alpha \in I$

Since each U_α is an open set therefore there exist $r > 0$

Such that $S_r(x) \subseteq U_\alpha$ for some $\alpha \in I$

$\Rightarrow \quad S_r(x) \subseteq \bigcup_{\alpha \in I} U_\alpha$

$\Rightarrow \quad \bigcup_{\alpha \in I} U_\alpha$ is an open set.

(ii) Let $\{ U_\alpha : \alpha = 1,2, \ldots \ldots n \}$ be finite collection of open sets in (X, d).

We are to prove that $\bigcap_{\alpha = 1}^{n} U_\alpha$ is an open set.

Let $x \in \bigcap_{\alpha = 1}^{n} U_\alpha$

$\Rightarrow \quad x \in U_\alpha \quad \forall \ \alpha = 1,2, \ldots \ldots n$

Since each U_α is an open set therefore there exist $r > 0$

Such that $S_{r_\alpha}(x) \subseteq U_\alpha \quad \forall \ \alpha = 1,2, \ldots \ldots n$

Let $r = min\{ r_1, r_2, r_3, \ldots \ldots r_n \}$

Then $S_r(x) \subseteq S_{r_\alpha}(x) \subseteq U_\alpha \quad \forall \ \alpha = 1,2, \ldots \ldots n$

$\Rightarrow \quad S_r(x) \subseteq U_\alpha \quad \forall \ \alpha = 1,2, \ldots \ldots n$

$\Rightarrow \quad S_r(x) \subseteq \bigcap_{\alpha = 1}^{n} U_\alpha$

$\Rightarrow \quad \bigcap_{\alpha = 1}^{n} U_\alpha$ is an open set.

(iii) To show that empty set ϕ is an open set, we have to show that each point in ϕ is the centre of some open sphere which is contained in ϕ. But since there is no point in ϕ, the condition is automatically satisfied.

Hence ϕ is an open set.

Since every open sphere centered at a point of X is contained in X.

$\therefore \quad X$ is an open set.

S11

Theorem

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An open sphere in a metric space (X, d) is an open set.

Proof

Let $S_r(x_\circ)$ be an open sphere in (X, d).

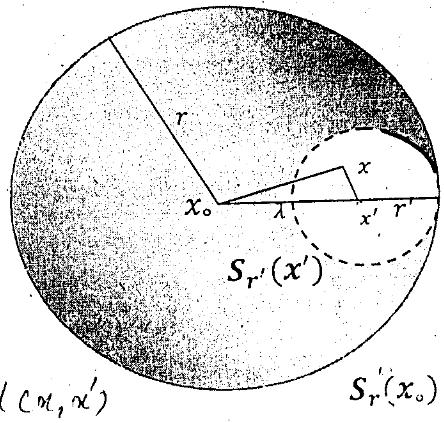
Let $x' \in S_r(x_\circ) \Rightarrow d(x', x_\circ) < r$

Let $d(x', x_\circ) = \lambda$

Let $r' = r - \lambda$, then $r' > 0$

We shall prove that $S_{r'}(x') \subseteq S_r(x_\circ)$

Let $x \in S_{r'}(x') \Rightarrow d(x, x') < r'$



Since (X, d) is a metric space,

$\therefore \quad d(x, x') + d(x', x_\circ) \geq d(x, x_\circ)$

$\Rightarrow \quad r' + \lambda > d(x, x_\circ) \qquad \because \lambda' > d(x, x')$

$\Rightarrow \quad r - \lambda + \lambda > d(x, x_\circ) \qquad \because r' = r - \lambda$

$\Rightarrow \quad d(x, x_\circ) < r$

$\Rightarrow \quad x \in S_r(x_\circ)$

33K8

Since $\quad x \in S_{r'}(x') \quad \Rightarrow \quad x \in S_r(x_\circ)$

$\therefore \quad S_{r'}(x') \subseteq S_r(x_\circ)$

Thus $S_r(x_\circ)$ is an open set.

S13

Hence open sphere in a metric space is an open set.

Theorem

A subset U of a metric space X is open if and only if U is union of open spheres.

Proof

Let (X, d) be a metric space. Let $U \subseteq X$. We have to prove that

U is an open set $\Leftrightarrow$ U is the union of open spheres.

We suppose that U is an open set. Since U is open therefore each point of U is the centre of some open sphere which is contained in U. Thus U is the union of open spheres.

Conversely suppose that U is the union of open spheres. Thus U is the union of open sets. ($\because$ Open spheres in metric space are open sets.)

Since the union of any number of open sets in a metric space is an open set. Thus U is an open set.

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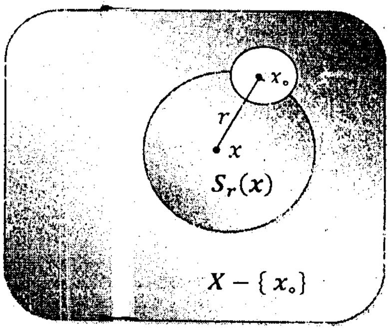
Theorem

Let X be a metric space and let $\{x_\circ\}$ be a singleton subset of X. Then $X - \{x_\circ\}$ is open.

Proof

Let $x \in X - \{x_\circ\}$

Let $d(x, x_\circ) = r$ --------- (1)



We shall prove that

$S_r(x) \subseteq X - \{x_\circ\}$

Let $x' \in S_r(x)$

$\Rightarrow \; d(x', x) < r$ ---------- (2)

From (1) and (2) we get

$\Rightarrow \; d(x', x) \neq d(x, x_\circ)$

$\Rightarrow \; d(x, x') \neq d(x, x_\circ)$ $\quad [\because d$ is a metric on X. So $d(x', x) = d(x, x')]$

$\Rightarrow \quad x' \neq x_\circ$

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$\Rightarrow \quad x' \notin \{x_\circ\}$

$\Rightarrow \quad x' \in X - \{x_\circ\}$

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Since $x' \in S_r(x) \;\; \Rightarrow \;\; x' \in X - \{x_\circ\}$

$\therefore \;\; S_r(x) \subseteq X - \{x_\circ\}$

Since every point x of $X - \{x_\circ\}$ is the centre of some open sphere contained in $X - \{x_\circ\}$.

Hence $X - \{x_\circ\}$ is an open set.

Question

Can a finite subset of a metric space be open?

Solution

We know that

(i) If $(X, d_\circ)$ is a discrete metric space, then every subset of X is open. Therefore a finite subset of a metric space is open.

(ii) If (R, d) is a usual metric space then $\{0\} \subseteq R$, is not open. Therefore a finite subset $\{0\}$ of R is not open.

Thus in general, we can say that, finite subset of a metric space may or may not open.

Metric Topology

The topology determined by a metric is called "metric topology".

Check in Past Paper

Theorem

If T is a collection of all open sets in a metric space (X, d), then T is a topology on X.

OR

A "metric space" is a topological space.

Proof

Let T be the collection of all open sets in a metric space (X, d).
We are to prove that, T is a topology on X.

(T_1) Let $U_\alpha \in T \quad \forall\ \alpha \in I$

$\Rightarrow$ U_α is an open set. $\forall\ \alpha \in I$

$\Rightarrow$ $\bigcup_{\alpha \in I} U_\alpha$ is open.

($\because$ Union of any number of open sets is open.)

$\Rightarrow$ $\bigcup_{\alpha \in I} U_\alpha \in T$

(T_2) Let $U_\alpha \in T \quad \forall\ \alpha = 1, 2, \ldots n$

$\Rightarrow$ $\bigcap_{\alpha = 1}^{n} U_\alpha$ is an open set.

($\because$ Intersection of finite number of open sets is open.)

$\Rightarrow$ $\bigcap_{\alpha = 1}^{n} U_\alpha \in T$ (By definition of T)

(T_3) Since ϕ, X both are open.

$\therefore$ $\phi, X \in T$ (By definition of T)

Thus T is a topology on X.

i.e. (X, T) is a topological space.

This shows that a "metric space" is a "topological space" whose topology is "metric topology".

Theorem

Every non-empty set can be given a metric topology.

Proof

We know that

(i) Every non-empty set can be given a metric and can be converted into metric space.

~~Therefore a finite subset of a metric space is open.~~

(ii) Every "metric space" is a "topological space" whose topology is a "metric topology".

Thus from (i) and (ii) we conclude that every non-empty set can be given a metric topology.

CLOSED SET

Closed Set

Let (X, d) be a metric space. Let $F \subseteq X$.

Then F is closed $\Leftrightarrow$ $F' = X - F$ is open

complement

Example

Let $X = R$ be the metric space and let $A = [a, b]$, where $a, b \in R$, & $a < b$. Show that A is closed set.

Solution

The given metric space is (R, d), where $d : R \times R \to R$ is given by

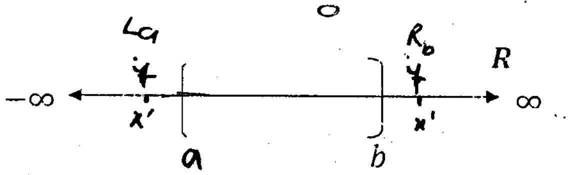
$$d(x_1, x_2) = |x_1 - x_2|$$

Since $A = [a, b]$

$A' = R - [a, b]$

$=]-\infty, a[\cup]b, \infty[$

$= L_a \cup R_b$



In order to prove A is closed; we will have to prove that A' is open.

Let $x' \in A' \quad \Rightarrow \quad x' \in L_a \cup R_b$

$\Rightarrow \quad x' \in L_a$ or $x' \in R_b$

Case-I If $x' \in L_a$ then $x' < a$

Let $d(x', a) = r$

$\Rightarrow \quad |x' - a| = r$

$\Rightarrow \quad x' - a = -r \quad \because x' < a$

$\Rightarrow \quad x' + r = a$ ------------------------------ (1)

Now $S_r(x') = \{\, x \mid x \in R,\ d(x, x') < r \,\}$

$= \{\, x \mid x \in R,\ |x - x'| < r \,\}$

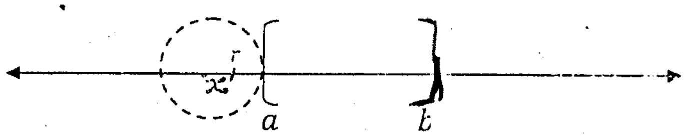
$= \{\, x \mid x \in R,\ x - x' < r,\ x - x' > -r \,\}$

$= \{\, x \mid x \in R,\ x < x' + r,\ x > x' - r \,\}$

$= \{\, x \mid x \in R,\ x' - r < x < x' + r \,\}$

$=]\, x' - r,\ x' + r \,[$

$=]\, x' - r,\ a \,[$ [By (1)]



Thus $S_r(x') =]\, x' - r,\ a \,[\subseteq L_a \subseteq L_a \cup R_b = A'$

i.e. $S_r(x') \subseteq A'$

Hence in this case A' is open.

Case-II If $x' \in R_b$ then $x' > b$

Let $d(x', b) = r$

$\Rightarrow \quad |x' - b| = r$

$\Rightarrow \quad x' - b = r \quad \because x' > b$

$\Rightarrow \quad x' - r = b$ ---------------------------- (2)

Now $S_r(x') = \{ x | x \in R, \ d(x, x') < r \}$

$= \{ x | x \in R, \ |x - x'| < r \}$

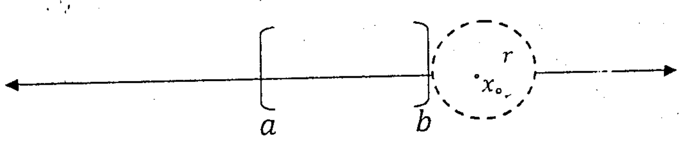
$= \{ x | x \in R, \ x - x' < r, \ x - x' > -r \}$

$= \{ x | x \in R, \ x < x' + r, \ x > x' - r \}$

$= \{ x | x \in R, \ x' - r < x < x' + r \}$

$=]\, x' - r, \ x' + r \,[$

$=]\, b, \ x' + r \,[$ [By (2)]



Thus $S_r(x') =]\, b, \ x' + r \,[\subseteq R_b \subseteq L_a \cup R_b = A'$

i.e. $S_r(x') \subseteq A'$.

Hence in this case A' is also open.

Since in both the cases A' is open. Therefore A is closed set.

Example

Let R^2 be the metric space.
Let $F = \{ (x, y) | (x, y) \in R^2, \ (x - a)^2 + (y - b)^2 \le 1 \}$.
Show that F is closed set.

Solution

Here given metric space is (R^2, d) where $d: R^2 \times R^2 \to R$ is given by $d[(x_1, y_1), (x_2, y_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Here $F = \{ (x, y) | (x, y) \in R^2, \ (x - a)^2 + (y - b)^2 \le 1 \}$

Thus $F' = \{ (x, y) | (x, y) \in R^2, \ (x - a)^2 + (y - b)^2 > 1 \}$

In order to prove that F is closed, we will show that F' is open.

Let $P' \in F'$. Let $d(P', P_0) = \lambda$

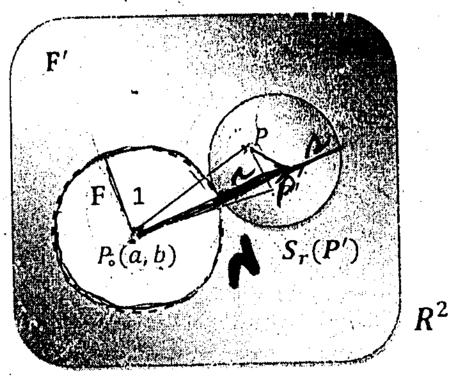
Let $r = \lambda - 1$, clearly $r > 0$

We shall prove that $S_r(P') \subseteq F'$

Let $P \in S_r(P') \Rightarrow d(P, P') < r$

Since d is a metric on R^2

$\therefore \ d(P', P) + d(P, P_0) \ge d(P', P_0)$



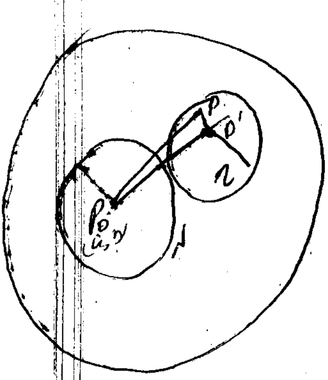
let $P \in F'$

$\Rightarrow \quad r + d(P,P_\circ) > \lambda$ $\quad \because \lambda > d(P,P')$ construct an open sphere centred at P and having radius r such that $d(P_0,P') = \lambda$

$\Rightarrow \quad d(P,P_\circ) > \lambda - r$

$\Rightarrow \quad d(P,P_\circ) > \underline{\lambda - (\lambda - 1) = 1}$

$\lambda - (\lambda - 1)$
$\lambda - \lambda + 1 = 1$



$\Rightarrow \quad d(P,P_\circ) > 1$

$\Rightarrow \quad P \in F'$

Since $P \in S_r(P') \Rightarrow P \in F'$ $\quad$ take a pt $P' \in S_r(P)$ such that $d(P,P') < r$

$\therefore \quad S_r(P') \subseteq F'$

$\Rightarrow \quad F'$ is an open set. $\quad$ Join the point P_0 with P & P'

$\Rightarrow \quad F$ is closed set. $\quad$ In this way we get vector triangle P_0PP'

Example

Let R^2 be the metric space. $\quad$ From triangle inequality

Let $A = \{ (x,y) | (x,y) \in R^2,\ x^2 + y^2 \le 1 \}$ be a subset of R^2.

Is A a closed set in R^2? $\quad d(P_0,P) \le d(P_0,P') + d(P',P)$

Solution

$d(P_0,P) \le \lambda + r$

Here given metric space is (R^2, d) where $d: R^2 \times R^2 \to R$ is given by $d[(x_1,y_1),(x_2,y_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Here $A = \{ (x,y) | (x,y) \in R^2,\ x^2 + y^2 \le 1 \}$ $\quad$ As $r = \lambda - 1$

Thus $A' = \{ (x,y) | (x,y) \in R^2,\ x^2 + y^2 > 1 \}$ $\quad d(P_0,P) \le \lambda + (\lambda - 1)$

In order to prove that A is closed, we will show that A is open.

$d(P_0,P) \le 1$

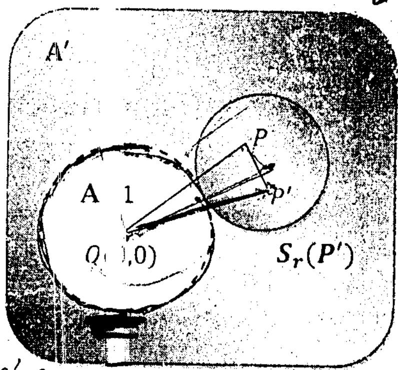
Let $P' \in A'$. Let $d(P',O) = \lambda$

Let $r = \lambda - 1$, clearly $r > 0$

We shall prove that $S_r(P') \subseteq A'$

Let $P \in S_r(P') \Rightarrow d(P,P') < r$

Since d is a metric on R^2



R^2

$P \in F'$
$P \in S_r(P')$
$F \subseteq S_r(P')$
F is closed

$\therefore \ d(P',P) + d(P,O) \ge d(P',O)$

$\Rightarrow \quad r + d(P,O) > \lambda$ $\quad \because \lambda > d(P',P)$

$\Rightarrow \quad d(P,O) > \lambda - r$

$\Rightarrow \quad d(P,O) > \lambda - (\lambda - 1) = 1$

$\Rightarrow \quad d(P,O) > 1$

$\Rightarrow \quad P \in A'$

Since $P \in S_r(P') \Rightarrow P \in A'$

$\therefore \quad S_r(P') \subseteq A'$

$\Rightarrow \quad A'$ is an open set.

$\Rightarrow \quad A$ is closed set.

Example

Let R be the real line and let $A = \{ x|x \in R,\ 0 \leq x < 1 \}$, be a subset of R. Show that A is not closed.

Solution

The given metric space is (R, d), where $d: R \times R \to R$ is given by

$$d(x_1, x_2) = |x_1 - x_2|$$

Here $\quad A = \{ x|x \in R,\ 0 \leq x < 1 \}$

$\quad\quad = [0,1[$

$\therefore \quad A' = R - A$

$\quad\quad =]-\infty, 0\, [\cup [\, 1, \infty\, [$

Note that, $1 \in A'$. We take $x_\circ = 1$, and $r > 0$

Then $S_r(x_\circ) = \{ x|x \in X,\ d(x, x_\circ) < r \}$

Put $x_\circ = 1$ and $X = R$

$S_r(1) = \{ x|x \in R,\ d(x, 1) < r \}$

$= \{ x|x \in R,\ |x - 1| < r \}$

$= \{ x|x \in R,\ x - 1 < r,\ x - 1 > -r \}$

$= \{ x|x \in R,\ x < 1 + r,\ x > 1 - r \}$

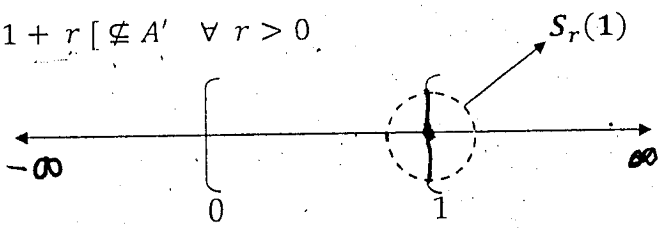
$= \{ x|x \in R,\ 1 - r < x < 1 + r \}$

$=]\, 1 - r,\ 1 + r\, [$

But $S_r(1) =]\, 1 - r,\ 1 + r\, [\not\subseteq A' \quad \forall\ r > 0$

Thus A' is not open.

$\Rightarrow \quad A$ is not closed.



Theorem Imp

A subset U of a metric space is open if and only if $X - U$ is closed.
A subset F " " " " " " F' closed

Proof

Let (X, d) be a metric space. We have to prove that

U is open $\Leftrightarrow$ $X - U$ is closed.

Suppose U is an open set.

Then $\quad (X - U)' = (U')' \quad\quad \because X - U = U'$

$\quad\quad = U \quad\quad$ (Open set)

Since $(X - U)'$ is an open set.

$\therefore \quad X - U$ is a closed set.

Conversely suppose that $X - U$ is a closed set.

Then $(X - U)'$ is an open set.

$\Rightarrow \quad (U')'$ is an Open set. $\quad \because \quad X - U = U'$

$\Rightarrow \quad U$ is an Open set.

Theorem

Let X be a metric space.

(i) Intersection of any collection $\{ F_\alpha : \alpha \in I\}$ of closed sets is closed.

(ii) Union of finite collection $\{ F_1, F_2, \ldots\ldots, F_n\}$ of closed set is closed.

(iii) X and ϕ are closed.

Proof

(i) Let $\{ F_\alpha : \alpha \in I \}$ be any collection of closed sets in (X, d). $(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$

Then $F_\alpha{}'$ is open. $\forall\ \alpha \in I$

$\Rightarrow\ \bigcup_{\alpha \in I} F_\alpha{}'$ is open. ($\because$ Union of any number of open sets is open)

$\Rightarrow\ \left(\bigcap_{\alpha \in I} F_\alpha\right)'$ is open. $\quad \because\ \bigcup_{\alpha \in I} F_\alpha{}' = \left(\bigcap_{\alpha \in I} F_\alpha\right)'$

$\Rightarrow\ \bigcap_{\alpha \in I} F_\alpha$ is closed. $\quad \bigcap_{\alpha \in I} F_\alpha = F_1 \cap F_2 \cap F_3 \cdots\cdots$

(ii) Let $\{F_\alpha : \alpha = 1,2,\ldots,n\}$ be any finite collection of closed sets in (X, d).

Then $F_\alpha{}'$ is open. $\forall\ \alpha = 1,2,\ldots,n$

$\Rightarrow\ \bigcap_{\alpha = 1}^{n} F_\alpha{}'$ is open. ($\because$ Intersection of finite number of open sets is open)

$\Rightarrow\ \left(\bigcup_{\alpha = 1}^{n} F_\alpha\right)'$ is open. $\quad \because\ \bigcap_{\alpha = 1}^{n} F_\alpha{}' = \left(\bigcup_{\alpha = 1}^{n} F_\alpha\right)'$

$\Rightarrow\ \bigcup_{\alpha = 1}^{n} F_\alpha$ is closed.

(iii) Since $\phi' = X - \phi = X$ which is open.

$\Rightarrow\ \phi$ is closed.

And $X' = X - X = \phi$ which is open.

$\Rightarrow\ X$ is closed.

Question

Is N closed in R?

Solution

Here $N = \{ 1,2,3, \ldots\ldots \}$

$N' = R - N$

$=]-\infty, 1[\ \cup\ ]1,2[\ \cup\ ]2,3[\ \cup \ldots\ldots$

$=$ Union of open intervals in R

$=$ Union of open sets ($\because$ An open intervals in R is an open set)

$=$ Open set ($\because$ Union of any number of open sets is an open set)

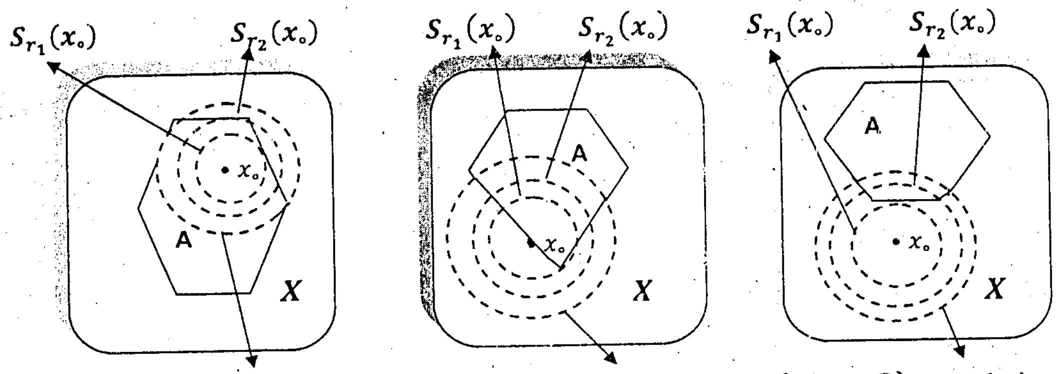
Since N' is an open set.

$\Rightarrow\ N$ is a closed set.

LIMIT POINT

Limit Point

Let (X,d) be a metric space. Let $A \subseteq X$ and $x_\circ \in X$. Then $x_\circ$ is called limit point of A if each open sphere centered at $x_\circ$ contains at least one point of A different from $x_\circ$.



(Fig – 1) $S_{r_3}(x_\circ)$ (Fig – 2) $S_{r_3}(x_\circ)$ (Fig – 3) $S_{r_3}(x_\circ)$

In Fig–1, $x_\circ$ is a limit point of A.

In Fig–2, $x_\circ$ is also a limit point of A.

In Fig–3, $x_\circ$ is not a limit point of A.

Theorem

Let (X,d) be a discrete metric space. Let $A \subseteq X$. Then A has no limit point.

Proof

Consider the discrete metric space $(X, d_\circ)$.

Here $d_\circ : X \times X \to R$ is defined by

$$d_\circ(x_1,x_2) = \begin{cases} 0 & if\ x_1 = x_2 \\ 1 & if\ x_1 \neq x_2 \end{cases}$$

We have to prove that, $A \subseteq X$ has no limit point.

We shall prove it by contradiction method.

Suppose $x_\circ \in X$ such that $x_\circ$ is a limit point of A.

Let $0 < r < 1$ then $S_r(x_\circ) = \{x_\circ\}$ ---------- (1)

$\because$ In a discrete metric space the open sphere with radius less than 1 is always singleton.

Here (1) shows that $S_r(x_\circ)$ contains no point of A different from $x_\circ$.

Thus $x_\circ$ is not a limit point of A.

Hence A has no limit point.

Question

Let R be the metric space. Let $A = \left\{x | x \in R,\ x = \frac{1}{n}, n \in N\right\}$ be a subset of R. Show that "0" is a limit point of A.

Solution

Here metric space is (R, d), where $d: R \times R \to R$ be defined by

$$d(x_1, x_2) = |x_1 - x_2|$$

Here $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\right\}$

Then $S_r(0) = \{x | x \in R,\ d(x, 0) < r\}; \quad r > 0$

$$= \{x | x \in R,\ |x - 0| < r\}$$

$$= \{x | x \in R,\ |x| < r\}$$

$$= \{x | x \in R,\ x < r,\ x > -r\}$$

$$= \{x | x \in R,\ -r < x < r\}$$

$$=]-r, +r[$$

Clearly for every $r > 0$, $S_r(0) =]-r, +r[$ contains a point of A different from "0".

Thus "0" is the limit point of A.

Question

Let R be the metric space. Let $A = \left\{x | x \in R, x = 1 \text{ or } x = 1 + \frac{1}{n}, n \in N\right\}$ be a subset of R. Show that "1" is a limit point of A.

Solution

Here metric space is (R, d), where $d: R \times R \to R$ be defined by

$$d(x_1, x_2) = |x_1 - x_2|$$

Here $A = \left\{x | x \in R, x = 1 \text{ or } x = 1 + \frac{1}{n}, n \in N\right\}$

$$= \{x | x \in R, x = 1\} \cup \{x | x \in R, x = 1 + 1/n, n \in N\}$$

$$= \{1\} \cup \left\{2, \frac{3}{2}, \frac{4}{3}, \ldots\ldots\right\}$$

$$= \left\{1, 2, \frac{3}{2}, \frac{4}{3}, \ldots\ldots\right\}$$

Now $S_r(1) = \{x | x \in R,\ d(x, 1) < r\}$

$$= \{x | x \in R,\ |x - 1| < r\}$$

$$= \{x | x \in R,\ x - 1 < r,\ x - 1 > -r\}$$

$$= \{x | x \in R,\ x < 1 + r,\ x > 1 - r\}$$

$$= \{x | x \in R,\ 1 - r < x < 1 + r\}$$

$$=]1 - r, 1 + r[$$

Clearly for every $r > 0$, $S_r(1) =]1 - r, 1 + r[$ contains a point of A different from "1".

Thus "1" is a limit point of A.

Question

Let R be the metric space. Let $A = \{ x | x \in R,\ 0 < x < 1 \}$ be a subset of R. Show that "0" and "1" are the limit point of A.

Solution

Here metric space is (R, d), where $d: R \times R \to R$ be defined by

$$d(x_1, x_2) = |x_1 - x_2|$$

Here $A = \{ x | x \in R,\ 0 < x < 1 \}$

$=]\,0,\ 1\,[$

(i) First we shall prove that "0" is the limit point of A.

Now $S_r(0) = \{x | x \in R,\ d(x, 0) < r\};\quad r > 0$

$= \{x | x \in R,\ |x - 0| < r\}$

$= \{x | x \in R,\ |x| < r\}$

$= \{x | x \in R,\ x < r,\ x > -r\}$

$= \{x | x \in R,\ -r < x < r\}$

$=]-r,\ +r[$

Clearly for every $r > 0$, $S_r(0) =]-r,\ +r\,[$ contains a point of A different from "0".

Thus "0" is a limit point of A.

(ii) Now we shall prove that "1" is the limit point of A.

Now $S_r(1) = \{ x | x \in R,\ d(x, 1) < r \}$

$= \{ x | x \in R,\ |x - 1| < r \}$

$= \{ x | x \in R,\ x - 1 < r,\ x - 1 > -r \}$

$= \{ x | x \in R,\ x < 1 + r,\ x > 1 - r \}$

$= \{ x | x \in R,\ 1 - r < x < 1 + r \}$

$=]\,1 - r,\ 1 + r\,[$

Clearly for every $r > 0$, $S_r(1) =]\,1 - r,\ 1 + r\,[$ contains a point of A different from "1".

Thus "1" is a limit point of A.

Question

Let R be the metric space. Describe the limit points of the followings.
(a) N (b) Z

Solution

Here metric space is (R, d), where $d: R \times R \to R$ be defined by

$$d(x_1, x_2) = |x_1 - x_2|$$

(a) Here $N = \{1, 2, 3, \ldots\}$

Let $a \in R$ be a limit point of N.

Then $a \in N$ or $a \notin N$

<u>Case - I</u> When $a \in N$

Then $S_r(a) = \{x | x \in R,\ d(x, a) < r\}; \qquad r > 0$

$= \{x | x \in R,\ |x - a| < r\}$

$= \{x | x \in R,\ x - a < r,\ x - a > -r\}$

$= \{x | x \in R,\ x < a + r,\ x > a - r\}$

$= \{x | x \in R,\ a - r < x < a + r\}$

$=]\,a - r,\ a + r\,[$

Clearly for every $r > 0$, $S_r(1) =]\,a - r,\ a + r\,[$ contains no point of N different from "a".

Thus "a" is not the limit point of N.

<u>Case - II</u> When $a \notin N$, we can also prove that "a" is not a limit point of N.

Thus N has no limit point.

(b) Here $Z = \{\ldots\ldots -3, -2, -1, 0, 1, 2, \ldots\ldots\}$

Let $a \in R$ be a limit point of Z.

Then $a \in Z$ or $a \notin Z$

<u>Case - I</u> When $a \in Z$

Then $S_r(a) = \{x | x \in R,\ d(x, a) < r\}; \qquad r > 0$

$= \{x | x \in R,\ |x - a| < r\}$

$= \{x | x \in R,\ x - a < r,\ x - a > -r\}$

$= \{x | x \in R,\ x < a + r,\ x > a - r\}$

$= \{x | x \in R,\ a - r < x < a + r\}$

$=]\,a - r,\ a + r\,[$

Clearly for every $r > 0$, $S_r(1) =]\,a - r,\ a + r\,[$ contains no point of Z different from "a".

Thus "a" is not the limit point of Z.

<u>Case - II</u> When $a \notin Z$, we can also prove that "a" is not a limit point of Z.

Thus Z has no limit point.

NEIGHBOURHOOD

$x_0 \in S_r(x_0) \subseteq N$

Neighbourhood

Let (X, d) be a metric space. Let $x_\circ \in X$. Let $N \subseteq X$. Then N is called a neighbourhood of $x_\circ$, if $\exists$ an open sphere $S_r(x_\circ)$ such that $x_\circ \in S_r(x_\circ) \subseteq N$.

Example

Let R be the usual metric space. Let $x_\circ = 0 \in R$. Show that $]-r, r[$, $]-r, r]$, $[-r, r[$, and $[-r, r]$, $(r > 0)$ is a neighbourhood of 0.

Solution

We know that,

In a usual metric space R, the open sphere is an open interval.

(i) Now $0 \in]-r, r[\subseteq]-r, r[$ Where $]-r, r[$ is an open sphere in R

$\Rightarrow$ $]-r, r[$ is a neighbourhood of "0".

(ii) Now $0 \in]-r, r[\subseteq]-r, r]$ Where $]-r, r[$ is an open sphere in R

$\Rightarrow$ $]-r, r]$ is a neighbourhood of "0".

(iii) Now $0 \in]-r, r[\subseteq [-r, r[$ Where $]-r, r[$ is an open sphere in R

$\Rightarrow$ $[-r, r[$ is a neighbourhood of "0".

(iv) Now $0 \in]-r, r[\subseteq [-r, r]$ Where $]-r, r[$ is an open sphere in R

$\Rightarrow$ $[-r, r]$ is a neighbourhood of "0".

A 10

Theorem

Let (X, d) be a metric space. Let $A \subseteq X$. Let $x_\circ$ be a limit point of A. Then every neighbourhood of $x_\circ$ contains infinitely many points of A.

Proof

Let N be a neighbourhood of $x_\circ$, then $\exists$ an open sphere $S_r(x_\circ)$ (where $r > 0$) such that

$$x_\circ \in S_r(x_\circ) \subseteq N \text{ ---------------- } (1)$$

We are to prove that N contains infinite points of A.

We prove it by contradiction method.

Suppose N contains finite points of A.

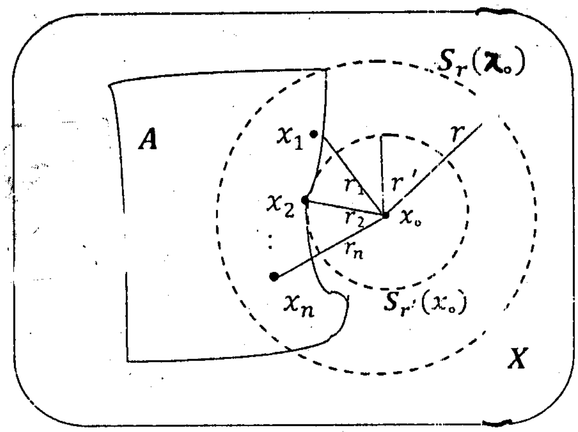
Then by (1) $S_r(x_\circ)$ also contains finite points of A.

Suppose $S_r(x_\circ)$ contains n points $x_1, x_2, x_3, \ldots\ldots, x_n$ of A.

Then $A \cap S_r(x_\circ) = \{x_1, x_2, x_3, \ldots\ldots, x_n\}$

Let $d(x_\circ, x_i) = r_i, \quad i = 1, 2, 3, \ldots\ldots, n$

Let $r' = min\,(r_1, r_2, r_3, \ldots\ldots, r_n)$



Clearly $S_{r'}(x_\circ)$ contains no point of A different from $x_\circ$.

This shows that, $x_\circ$ is not a limit point of A. This is a contradiction.

Hence N contains infinitely many points of A

"Functional Analysis" 1

CHAPTER No.1 { Normed Linear spaces }

Def: (1.1): **Norm:** A norm on a linear space X is a real valued function $\|\cdot\|$ (ie $\|\cdot\| : X \to \mathbb{R}$) whose value at x, denoted by $\|x\|$, have the following properties.

ⓐ $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$; $\forall\ x_1, x_2 \in X$.

ⓑ $\|\alpha x\| = |\alpha|\ \|x\|$; for any scaler α and $x \in X$.

ⓒ $\|x\| \geq 0$; $\forall\ x \in X$.

ⓓ $\|x\| = 0$ iff $x = 0$; $\forall\ x \in X$.

The pair $(X, \|\cdot\|)$ is called a normed linear space or normad vector space.

Remark (1.2): If x is a vector, its length is $\|x\|$, the length $\|x_1 - x_2\|$ of the vector difference $x_1 - x_2$ is the distance b/w the end points of the vectors x_1 and x_2.

Examples (1.3):

① the real linear space $\mathbb{R}$ is a normed linear space with norm $\|\cdot\| : \mathbb{R} \to \mathbb{R}$ defined by $\|x\| = |x|$; $\forall\ x \in \mathbb{R}$.

Pf: ⓐ For any $x_1, x_2 \in \mathbb{R}$, we have

$$\begin{aligned} \|x_1 + x_2\| &= |x_1 + x_2| \quad \text{(by def:)} \\ &\leq |x_1| + |x_2| \\ &= \|x_1\| + \|x_2\| \end{aligned}$$

ie $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$ $\forall\ x_1, x_2 \in \mathbb{R}$.

(b) For any scaler α and $x \in \mathbb{R}$, we have

$$\|\alpha x\| = |\alpha x| = |\alpha||x| = |\alpha|\,\|x\|$$

(c) For any $x \in \mathbb{R}$, we have:

$$\|x\| = |x| \geq 0 \Rightarrow \|x\| \geq 0.$$

(d) For any $x \in \mathbb{R}$, we have:

$$\|x\| = |x|$$

Thus $\|x\| = 0 \Leftrightarrow |x| = 0 \Leftrightarrow x = 0$.

ie $\|x\| = 0 \Leftrightarrow x = 0$.

(2) The complex linear space $\mathbb{C}$ is a normed linear space with the norm defined by:

$$\|z\| = |z| \;;\; \forall\, z \in \mathbb{C}.$$

Pf: (a) For any $z_1, z_2 \in \mathbb{C}$, we have:

$$\|z_1 + z_2\| = |z_1 + z_2| \quad \text{(by definition)}$$
$$\leq |z_1| + |z_2| \quad \text{(property of complex nos:)}$$
$$= \|z_1\| + \|z_2\| \quad \text{(by def:)}$$

ie $\|z_1 + z_2\| \leq \|z_1\| + \|z_2\| \;;\; \forall\, z_1, z_2 \in \mathbb{C}$.

(b) For any scaler α and $z \in \mathbb{C}$, we have:

$$\|\alpha z\| = |\alpha z| \quad \text{(by def:)}$$
$$= |\alpha||z| \quad \text{(property of complex nos:)}$$
$$= |\alpha|\,\|z\| \quad \text{(by def:)}$$

(c) For any $z \in \mathbb{C}$, we have:

$\|z\| = |z| = 0$ iff $z = 0$.

(d) For any $z \in \mathbb{C}$, we have:

$$\|z\| = |z| \geq 0. \quad \text{(property of complex nos:)}$$

ie $\|z\| \geq 0 \;;\forall\, z \in \mathbb{C}$.

Hence the complex linear space $\mathbb{C}$ is a normed linear space with the norm defined above.

* * * * * * * * * * * * * * * * * * *

(3) The spaces $\mathbb{R}^n$ (n-dimensional Euclidean space) and $\mathbb{C}^n$ (n-dimensional unitary space) of all n-tuples of real and complex numbers are normal linear spaces with the norms defined by:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \quad ; \qquad 1 \leq p \leq \infty$$

$$\text{i.e. } \|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p} \longrightarrow (1)$$

where $x_i = (x_1, x_2, \ldots, x_n)$

OR $\|x\| = \max\{|x_i| \;;\; i = 1, 2, \ldots, n\}$.

$$= \max\{|x_1|, |x_2|, \ldots, |x_n|\} \longrightarrow (2)$$

where $x_i = (x_1, x_2, \ldots, x_n)$, $\quad 1 \leq i \leq n$

Note: we can define more than one norm on a linear space.

Next, we introduce some special normed linear spaces.

(4) $l^p(x)$, when $\mathbb{C}^n$ or $\mathbb{R}^n$ is considered as normed linear spaces with the norm (1) of Example (3); we denote the space by $l^p(x)$.

Notice that we shall use $l^p(x)$ for both the

moreover, the question of whether the space under discussion is real or complex will either be clear from the context or we shall make a specific statement if necessary.

(5) $l^p = l_p$ = the space of all sequences $x = \{x_n\}$ with $\sum_{i=1}^{\infty} |x_i|^p < \infty$, $p \geq 1$; then this space l^p is a n.l.s with the norm

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \;;\; \forall\, x \in l^p$$

(6) $l^{\infty} = l_{\infty}$ is the space of all bounded sequences $x = \{x_i\}$, then l^{∞} is a n.l.s with the norm;

$$\|x\|_{\infty} = \sup |x_i| \quad ; \quad 1 \le i \le \infty$$
$$= \sup \{ |x_1|, |x_2|, \ldots \}.$$

(7) $C[a,b]$ is the space of all continuous real valued functions defined on $[a,b]$ ie $f : [a,b] \longrightarrow \mathbb{R}$, which is continuous.

Then $C[a,b]$ is a n.l.s with norms:

(i) $\|f\| = \sup |f(x)| \; ; \; \forall \, f \in C[a,b], \, x \in [a,b]$.

(ii) $\|f\| = \int_a^b |f(x)| \, dx \; ; \; \forall \, f \in C[a,b]$.

(8) C = This is the space of all convergent sequences in l^{∞}.

C_0 = This is also the space of all sequences in l^{∞} converging to zero.

then C and C_0 are normed linear spaces with norm as in l^{∞}.

Note that $C_0 \subset C \subset l^{\infty}$.

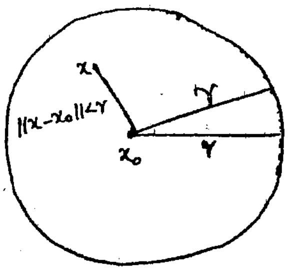
Definition (1.4):- Let X be a normed linear space.

(a) An open sphere (or open ball) with centre x_0 and radius $r > 0$ is the set:

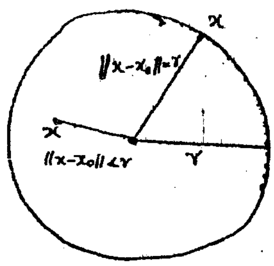
$$B(x_0 ; r) = \{ x \in X : \|x - x_0\| < r \}$$

A closed sphere (or ball) with centre x_0 and radius $r > 0$ is the set:

$$\bar{B}(x_0 ; r) = \{ x \in X : \|x - x_0\| \le r \}$$



$B(x_0; r)$

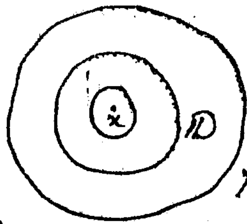


$\bar{B}(x_0; r)$

By the surface (or boundary) of this ball, we mean the set:

$$S(x_0; r) = \{x \in X : \|x - x_0\| = r\}$$

(b) A set D in X is said to be open if for every $x \in D$, there exists a ball with centre x ~~and~~ which is contained in D.



(c) A set D in X is said to be closed if for any sequence $\{x_n\}$ in D with $x_n \to x$ implies that $x \in D$.

(d) A set D is said to be bounded in X if there exists a constant M such that

$\|x\| \leq M \; ; \; \forall \; x \in D$.

(e) A set D is said to be compact if whenever $\{x_n\}$ is in D, there exists a cgt subsequence of $\{x_n\}$ whose limit is in D.

(f) A sequence $\{x_n\}$ is called bounded, if there exists a real constant $K > 0$ such that

$\|x_n\| \leq K \;\; \forall \; n$.

Proposition (1.5):

ⓐ Every norm linear space X is a metric space w.r.t The metric $d(x,y) = \|x-y\|$; $\forall x, y \in X$.

ⓑ $\big| \|x\| - \|y\| \big| \leq \|x-y\|$; $\forall x, y \in X$.

Proof: ⓐ let X be a norm linear space. Define a mapping $d : X \times X \to \mathbb{R}$ by:

$$d(x,y) = \|x-y\| \; ; \; \forall x, y \in X.$$

we show That d is a metric on X.

Since (i) $d(x,y) = \|x-y\| \geq 0$ (by def:)

ie $d(x,y) \geq 0$.

(II) $d(x,y) = \|x-y\| = 0$ iff $x-y=0$ (by def:)

iff $x = y$

ie $d(x,y) = 0$ iff $x = y$.

(iii) $d(x,y) = \|x-y\| = \|y-x\| = d(y,x)$

ie $d(x,y) = d(y,x)$.

(iv) $d(x,z) = \|x-z\| = \|x-y+y-z\| \leq \|x-y\| + \|y-z\|$
$= d(x,y) + d(y,z)$

so $d(x,z) \leq d(x,y) + d(y,z)$.

Hence d is a metric on norm linear space X, known as metric induced by a norm and hence X with d is a metric space.

ⓑ $\big| \|x\| - \|y\| \big| \leq \|x-y\|$; $\forall x, y \in X$.

Pf: we can write : $x = x-y+y$

$\Rightarrow \|x\| = \|x-y+y\| \leq \|x-y\| + \|y\|$

$\Rightarrow \|x\| - \|y\| \leq \|x-y\|. \longrightarrow (1)$

similarly we can write : $y = y-x+x$.

$\Rightarrow \|y\| = \|y-x+x\| \leq \|y-x\| + \|x\|$

$\Rightarrow -\|y-x\| \leq \|x\| - \|y\|$

$\Rightarrow -\|x-y\| \leq \|x\| - \|y\| \longrightarrow ②$

Combining ① and ②, we have:

$$-\|x-y\| \leq \|x\| - \|y\| \leq \|x-y\|.$$

$$\Rightarrow \big|\|x\| - \|y\|\big| \leq \|x-y\|.$$

Definition (1.6):

let X be a normed linear space and let $\{x_n\}$ be a sequence in X. Then

ⓐ we say that the sequence $\{x_n\}$ of elements of X converges to the limit $x \in X$ if for every $\varepsilon > 0$, there exists a +ve integer N such that

$\|x_n - x\| < \varepsilon$ for $n \geq N$.

In other words, we say that $\{x_n\}$ is convergent to the limit $x \in X$ iff $\lim\limits_{n\to\infty} \|x_n - x\| = 0$.

Symbolically we write $x_n \to x$ as $n \to \infty$.

ⓑ we say that the sequence $\{x_n\}$ in X is a Cauchy sequence if for every $\varepsilon > 0$, there exists a +ve integer N such that:

$\|x_m - x_n\| < \varepsilon$ for $m, n \geq N$.

In other words, $\{x_n\}$ is a Cauchy sequence iff $\lim\limits_{\substack{m\to\infty \\ n\to\infty}} \|x_m - x_n\| = 0$.

Exercise (1.7): Let X be a norm linear space.

1. If the limit of a sequence $\{x_n\}$ in X exists Then it is unique.
2. Every Convergent sequence in X is a Cauchy sequence, but the Converse is not true, in general.
3. A Cauchy sequence is Convergent iff it has a Convergent subsequence.
4. Every Cauchy sequence in X is bounded.

—*——*——*——*——*——*——*

Proposition (1.8): Let X be a norm linear space

(a) Norm is a Continuous function

ie if $x_n \to x$ Then $\|x_n\| \to \|x\|$

OR If $\{x_n\}$ is a Convergent sequence in X, Then $\|x_n\|$ is a Convergent sequence in $\mathbb{R}$.

(b) Addition and scaler multiplication are jointly Continuous in X ie If $x_n \to x$ and $y_n \to y$ Then $x_n + y_n \to x + y$.

and if $x_n \to x$ and $\alpha_n \to \alpha$, Then $\alpha_n x_n \to \alpha x$.

(c) If $x_n \to x$ and $y_n \to y$, Then $ax_n + by_n \to ax + by$ where a and b are Constants.

Proof: (a) Since $x_n \to x$. So by definition:

$$\lim_{n\to\infty} \|x_n - x\| = 0 \quad \text{(1)}$$

Now $\Big| \|x_n\| - \|x\| \Big| \le \|x_n - x\|$ [using (1.5) (b)]

so $\lim_{n\to\infty} \Big| \|x_n\| - \|x\| \Big| \le \lim_{n\to\infty} \|x_n - x\| = 0$ (by (1))

Thus $\|x_n\| \to \|x\|$ ie norm is a Continuous function.

—*——*——*——*——*——*——*——*

(b) Since $x_n \to x$ and $y_n \to y$. So by definition

$$\lim_{n\to\infty} \|x_n - x\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_n - y\| = 0$$

Now
$$\|(x_n + y_n) - (x+y)\| = \|x_n - x + y_n - y\|$$
$$\leq \|x_n - x\| + \|y_n - y\|.$$

So
$$\lim_{n\to\infty} \|(x_n+y_n) - (x+y)\| \leq \lim_{n\to\infty} \|x_n - x\| + \lim_{n\to\infty} \|y_n - y\|$$
$$= 0 + 0 \quad (\text{from above}).$$
$$= 0$$

Hence $x_n + y_n \longrightarrow x + y$

Next we show that $\alpha_n x_n \longrightarrow \alpha x$.

Since $\alpha_n \to \alpha$, so by definition; we have

$$\lim_{n\to\infty} |\alpha_n - \alpha| = 0$$

Now
$$\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\|$$
$$\leq \|\alpha_n x_n - \alpha_n x\| + \|\alpha_n x - \alpha x\|.$$
$$= \|\alpha_n (x_n - x)\| + \|x(\alpha_n - \alpha)\|$$
$$= |\alpha_n| \|x_n - x\| + |\alpha_n - \alpha| \|x\|.$$

So
$$\lim_{n\to\infty} \|\alpha_n x_n - \alpha x\| \leq \lim_{n\to\infty} |\alpha_n| \|x_n - x\| + \lim_{n\to\infty} |\alpha_n - \alpha| \|x\|.$$
$$= |\alpha_n| \lim_{n\to\infty} \|x_n - x\| + \lim_{n\to\infty} |\alpha_n - \alpha| \|x\|.$$
$$= 0 + 0 \quad (\text{from above})$$
$$= 0$$

Hence $\alpha_n x_n \longrightarrow \alpha x$

ie scalar multiplication and addition are jointly continuous.

——— * ——— * ——— * ——— * ——— * ——— * ———

(c) Since $x_n \to x$ and $y_n \to y$, so we have:

$$\lim_{n\to\infty} \|x_n - x\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_n - y\| = 0.$$

Now $\|(ax_n + by_n) - (ax+by)\| = \|ax_n - ax + by_n - by\|$

$\leq \|ax_n - ax\| + \|by_n - by\|$

Now

$$\lim_{n\to\infty} \|(ax_n + by_n) - (ax+by)\| \leq \lim_{n\to\infty}\left(\|ax_n - ax\| + \|by_n - by\|\right)$$

$$= \lim_{n\to\infty} \|a(x_n - x)\| + \lim_{n\to\infty} \|b(y_n - y)\|$$

$$= |a| \lim_{n\to\infty} \|x_n - x\| + |b| \lim_{n\to\infty} \|y_n - y\|$$

$$= 0 + 0 \text{ (from above)}$$
$$= 0$$

Thus $ax_n + by_n \longrightarrow ax + by$

which completes the proof.

Bounded Linear operators:

Before defining a bounded linear operator, we recall ~~some~~ definitions and results from "Algebra".

Definition: Let X and Y be linear spaces with the same scaler field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$).
Let A be a function with $D(A)$ in X and range $R(A)$ in Y [ie $A : D(A) \subset X \longrightarrow R(A) \subset Y$]
Then A is called a linear operator if $D(A)$ is a subspace of X and if:

$$\left.\begin{array}{ll} \text{(a)} & A(x_1 + x_2) = Ax_1 + Ax_2 \; ; \forall\, x_1, x_2 \in D(A) \\ \text{(b)} & A(\alpha x) = \alpha A(x) \; ; \forall\, \alpha \in \mathbb{K} \text{ and } x \in D(A) \end{array}\right\} \to (1)$$

Clearly Condition (1) is equivalent to:

$$A(\alpha x_1 + \beta x_2) = \alpha A x_1 + \beta A x_2 \; ; \forall\, \alpha, \beta \in \mathbb{K} \text{ and } x_1, x_2 \in D(A).$$

If $D(A) = X$, we often say that A is a linear operator on (or from) X to Y.

Remark: ① It follows immediately by induction from ⓐ and ⓑ of above definition that

$$A(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) = \alpha_1 A x_1 + \alpha_2 A x_2 + \cdots + \alpha_n A x_n.$$

② If $\alpha = 0$ in above definition, then we have $A(0) = 0$.

③ An important subset of the domain of A is the null space of A denoted by $\mathcal{N}(A)$ and is defined by:

$$\mathcal{N}(A) = \{x \in D(A) : Ax = 0\}.$$

It is readialy verified that $\mathcal{N}(A)$ is a subspace.
[illegible], the $Ax_1 = 0, Ax_2 = 0$. Let α, β [illegible]
[illegible] $= \ldots \Rightarrow \alpha x_1 + \beta x_2 \in \mathcal{N}(A)$

Examples:—

① The identity operator $I : X \longrightarrow X$ defined by $I(x) = x$; $\forall x \in X$ is clearly a linear operator from X into itself.

② Zero operator $T : X \longrightarrow Y$ defined by: $T(x) = 0$; $\forall x \in X$ is clearly linear operator. Note that a zero operator is also called Null operator OR Trivial operator.

③ Consider the linear space P of all polynomials $p(x)$ with real coefficients, defined on $[0,1]$. Then the mapping D defined by: $D(p) = \frac{dp}{dx}$, is a linear operator from P into itself.

④ The mapping T defined by: $T(f) = \int_0^1 f(x)\,dx$ is clearly seen to be a linear operator of $C[0,1]$, the space of continuous real functions defined on the closed unit interval $[0,1]$ into the real linear space of all real nos: ie $T : C[0,1] \longrightarrow \mathbb{R}$.

—— * —— * —— * —— * —— * —— * ——

Definition: (a) A mapping $T: D(T) \subset X \longrightarrow Y$ is said to be injective or one-to-one if different points ~~in~~ the domain has different images.

ie if for any $x_1, x_2 \in D(T)$, we have:

$x_1 \neq x_2 \Rightarrow Tx_1 \neq Tx_2$

or equivalently $Tx_1 = Tx_2 \Rightarrow x_1 = x_2$.

(b) A mapping $T: D(T) \subset X \longrightarrow Y$ is said to be surjective or onto if $R(T) = Y$ ie if every element of Y is the image of at least one element in X.

(c) If T is both injective and surjective, Then it is called bijective.

Notations: If a linear operator A has an inverse, Then it is denoted by A^{-1}. The statement "A^{-1} exists" is the same as "A has an inverse".

It is known that A^{-1} exists iff A is one-to-one

ie $Ax_1 = Ax_2 \Longrightarrow x_1 = x_2$.

Thm (A): let A be a linear operator, Then A^{-1} exists iff $Ax = 0 \Rightarrow x = 0$.

when A^{-1} exists, Then A^{-1} is a linear operator.

Thm (B): If A is a linear operator from a linear space X into a linear space Y. Then A^{-1} exists iff A is one-to-one and onto.

Theorem: Let A be a linear operator, then A^{-1} exists iff $Ax=0 \Rightarrow x=0$. When A^{-1} exists, it is also a linear operator.

Proof: Before proving the above result, we remember the following fact:

"The inverse of an operator A exists iff A is one-to-one ie if $Ax_1 = Ax_2 \Rightarrow x_1 = x_2$; $\forall x_1, x_2 \in D(A)$."

Now we prove the required result.

First let us suppose that A^{-1} exists. Suppose x is an arbitrary vector in $D(A)$ such that $Ax=0$.

But as A is a linear operator, so that $A(0)=0$

i.e. $Ax = A(0)$. But A^{-1} exists, so A is one-to-one

therefore $Ax = A(0) \Rightarrow x=0$.

Conversely, let us suppose that $Ax=0 \Rightarrow x=0$.

We are to prove that A^{-1} exists and for this we will show that A is one-to-one.

For this let $Ax_1 = Ax_2$, where $x_1, x_2 \in D(A)$

$\Rightarrow Ax_1 - Ax_2 = 0 \Rightarrow A(x_1 - x_2) = 0$ (as A is linear)

$\Rightarrow x_1 - x_2 = 0$ [by supposition]

$\Rightarrow x_1 = x_2$

which shows that A is one-to-one.

Consequently A^{-1} exists. Hence proved.

Finally we show that when A^{-1} exists, then it is also a linear operator.

Now let $x_1, x_2 \in D(A)$, then we can find y_1, y_2 in $R(A)$ such that $Ax_1 = y_1$ and $Ax_2 = y_2$

Since A^{-1} exists, so that $x_1 = A^{-1}y_1$ and $x_2 = A^{-1}y_2$.

Now $y_1 + y_2 = Ax_1 + Ax_2 = A(x_1 + x_2)$ [$\because A$ is Linear]

Since A^{-1} exists, so $A^{-1}(y_1+y_2) = x_1 + x_2 = A^{-1}(y_1) + A^{-1}(y_2)$

Again let $\alpha \in \mathbb{K}$ and Consider αy_1.

Now $\alpha y_1 = \alpha (Ax_1) = A(\alpha x_1)$ [$\because A$ is linear]

$\Rightarrow A^{-1}(\alpha y_1) = \alpha x_1 = \alpha A^{-1}(y_1)$ [$\because A^{-1}$ exists]

$\Rightarrow A^{-1}(\alpha y_1) = \alpha A^{-1}(y_1)$

Hence A^{-1} is also a linear operator.

<u>Theorem</u>: A^{-1} exists iff $N(A)=\{0\}$, when A is linear operator.

<u>Proof</u>: First we recall that A^{-1} exists iff A is one-one. Now suppose that A^{-1} exists, we prove that $N(A)=\{0\}$. For this let $x \in N(A)$, so by def:, $Ax=0$. But as A is a linear operator, so $A(0)=0$ therefore $Ax = A(0)$. Since A^{-1} exists, so A is one-one. Hence $x=0$. Therefore $N(A)=\{0\}$.

Conversely suppose that $N(A)=\{0\}$ and we show that A^{-1} exists and to show that A^{-1} exists, we show that A is one-to-one.

For this let $Ax_1 = Ax_2$, where $x_1, x_2 \in D(A)$

$\Rightarrow Ax_1 - Ax_2 = 0$

$\Rightarrow A(x_1 - x_2) = 0$ [$\because A$ is linear]

$\Rightarrow x_1 - x_2 \in N(A) = \{0\}$

$\Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$

Thus A is one-to-one, Consequently A^{-1} exists. This completes the required proof.

Definition (1.9): Let X and Y be two normed linear spaces over a field $\mathbb{K}$ and $T: X \to Y$ be a linear operator, Then

(a) we say that T is continuous at $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that
$\|Tx - Tx_0\| < \epsilon$ whenever $\|x - x_0\| < \delta$.

(b) we say that T is continuous on X if it is continuous for every point of X.

OR T is continuous on X iff for any sequence $\{x_n\}$ in X with $x_n \to x$ implies $Tx_n \to Tx$.

(c) T is continuous at the origin iff $x_n \to 0$ implies $Tx_n \to 0$.

(d) we say that T is uniformly continuous on X if for ~~every~~ any $x_1, x_2 \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that:
$\|Tx_1 - Tx_2\| < \epsilon$ whenever $\|x_1 - x_2\| < \delta$.

Proposition (1.10): (a) A uniformly continuous function is continuous.

(b) A continuous function on a compact space is uniformly continuous.

Definition (1.11): An operator $T: X \to Y$ is said to be bounded if there exists a constant $M > 0$ such that $\|Tx\| \leq M\|x\|$; $\forall x \in X$.

Theorem (1.12) Let $T: X \to Y$ be a linear operator from a n.l. space X into a n.l.s Y; Then

(a) If T is Continuous at some point $x_0 \in X$, then T is uniformly continuous.

(b) T is (uniformly) Continuous iff T is bounded.

Proof: (a) let T be continuous at some point $x_0 \in X$, then by definition, for every $\varepsilon > 0$ there exists $\delta > 0$ such that:

$\|Tx - Tx_0\| < \varepsilon$ whenever $\|x - x_0\| < \delta$ ↳ (1)

let y_1, y_2 be any two points in X.

let $w = y_1 - y_2 + x_0$, Then $w \in X$ because X is a linear space (closed under addition).

suppose $\|w - x_0\| < \delta$, Then by (1), we have:

$\|Tw - Tx_0\| < \varepsilon$

ie $\|y_1 - y_2 + x_0 - x_0\| < \delta$ implies $\|T(y_1 - y_2 + x_0) - Tx_0\| < \varepsilon$

ie $\|y_1 - y_2\| < \delta$ implies $\|Ty_1 - Ty_2 + Tx_0 - Tx_0\| < \varepsilon$. ($\because T$ is linear operator)

ie $\|Ty_1 - Ty_2\| < \varepsilon$ whenever $\|y_1 - y_2\| < \delta$.

Thus T is uniformly continuous on X.

Note: The Converse of this result is also true, because by proposition (1.10 (a)), we have: " Every unif. Continuous function is Continuous".

[b] suppose that T is bounded. So by definition there exists a constant $M > 0$ such that

$\|Tx\| \le M\|x\|$, $\forall x \in X$.

Now consider any point $x_0 \in X$. Let $\epsilon > 0$ be given. Then for every $x \in X$ such that $\|x - x_0\| < \delta$ where $\delta = \epsilon/M$, we have:

$$\begin{aligned} \|Tx - Tx_0\| &= \|T(x - x_0)\| \quad (\because T \text{ is linear}) \\ &\le M\|x - x_0\| \quad (\because T \text{ is bounded}) \\ &\le M \cdot \delta \\ &= M \cdot \epsilon/M \\ &= \epsilon \end{aligned}$$

ie $\|Tx - Tx_0\| < \epsilon$ whenever $\|x - x_0\| < \delta$

Since x_0 was an arbitrary point of X, this result shows that T is continuous on X.

$\Rightarrow$ T must be continuous at some point of X.

Therefor by part ⓐ, it is uniformly continuous.

Conversely, if T is continuous at origin, then there exits $\delta > 0$ such that:

$$\|Tu\| \le 1 \text{ if } \|u\| \le \delta \quad (\because T0 = 0)$$

Given any $x \in X$, we may write:

$$\underset{\boxed{\because X \text{ is L.S}}}{x = cu} \;, \text{ where } \|u\| = \delta \text{ and } c = \delta^{-1}\|x\| > 0$$

ie c is Constt.

Then $Tx = T(cu) \Rightarrow \|Tx\| = \|T(cu)\| = c\|Tu\|$

$$\begin{aligned} &\le c \quad (\because \|Tu\| \le 1) \\ &= \delta^{-1}\|x\| \end{aligned}$$

If we put $m = \delta^{-1}$, then we have,

$\|Tx\| \le m\|x\|$ $\forall x \in X$, which shows that T is bounded.

Proof (b): Suppose that T is continuous on X, then the statement "T is continuous at some point of X" is obviously true.

Conversely, suppose that T is continuous at some point $x_0 \in X$, then by definition for every $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$\|Tx - Tx_0\| < \varepsilon \text{ whenever } \|x - x_0\| < \delta, \; \forall x \in X. \longrightarrow (i)$$

We show that T is continuous on X.

For this let y be any arbitrary point of X, then we can write: $x - y = (x - y + x_0) - x_0$

Clearly $x - y + x_0 \in X$ ($\because X$ is a linear space)

Now

Since the condition (i) is true $\forall x \in X$ and since $x - y + x_0 \in X$; so by (i), we can write:

$$\|T(x - y + x_0) - Tx_0\| < \varepsilon \quad W \quad \|(x - y + x_0) - x_0\| < \delta$$

$$\Rightarrow \|Tx - Ty + Tx_0 - Tx_0\| < \varepsilon \quad W \quad \|x - y + x_0 - x_0\| < \delta$$

$$\Rightarrow \|Tx - Ty\| < \varepsilon \quad W \quad \|x - y\| < \delta$$

$\Rightarrow$ T is continous at y. But y was an arbitrary point of X, so T is continuous on every point of X, consequently T is continuous on X.

Pf: (c):- Suppose that T is bounded, then by definition there exists a +ve constant M such that:

$$\|Tx\| \leq M\|x\| \; ; \; \forall x \in X \longrightarrow (*)$$

We show that T is continuous on X.

For this let $\{x_n\}$ be a sequence in X such that $x_n \to x$. In order to show that T is continuous on X, we show that $Tx_n \to Tx$.

Now since $\{x_n\}$ is a sequence in X, so by condition (*), we have:

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<u>Theorem (1.13)</u>: let X and Y be norm linear spaces and $T: X \to Y$ be a linear operator, then:

ⓐ T is continuous on X iff it is uniformly contin on X.

ⓑ T is contin: on X iff it is continuous at some point of X.

ⓒ T is continuous on X iff it is bounded.

<u>Proof</u>: ⓐ suppose that T is continuous on X, then it is continuous at every point of X.

let $x_0 \in X$, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|T(x) - T(x_0)\| < \varepsilon$ whenever $\|x - x_0\| < \delta$; (i)

we shall show that T is uniformly continuous on X.

For this let x_1, x_2 be any two points of X and let $w = x_1 - x_2 + x_0$, then $w \in X$ ($\because X$ is a linear space)

so Replacing x by w in (i), we get:

$$\|T(w) - T(x_0)\| < \varepsilon \quad \text{whenever } \|w - x_0\| < \delta$$

$$\text{ie } \|T(x_1 - x_2 + x_0) - T(x_0)\| < \varepsilon \quad \text{W } \|x_1 - x_2 + x_0 - x_0\| < \delta$$

$$\text{ie } \|Tx_1 - Tx_2 + Tx_0 - Tx_0\| < \varepsilon \quad \text{W } \|x_1 - x_2\| < \delta$$

$$\text{ie } \|Tx_1 - Tx_2\| < \varepsilon \quad \text{W } \|x_1 - x_2\| < \delta$$

which shows that T is uniformly continuous on X.

conversely, suppose that T is uniformly continuous on X. then by definition, for every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\|Tx_1 - Tx_2\| < \varepsilon \quad \text{W } \|x_1 - x_2\| < \delta \; ; \; \forall x_1, x_2 \in X$$

$\Rightarrow T$ is continuous at $x_2 \in X$. But $x_2 \in X$ is an arbitrary point of X, so T is continuous on X. this completes the proof.

$$\|T(x_n - x)\| \le M\|x_n - x\|$$

$\Rightarrow \|Tx_n - Tx\| \le M\|x_n - x\|$ ($\because$ T is linear)

$$\Rightarrow \operatorname*{Lt}_{n\to\infty} \|Tx_n - Tx\| \le \operatorname*{Lt}_{n\to\infty} M\|x_n - x\|$$

$$= M \operatorname*{Lt}_{n\to\infty} \|x_n - x\|$$

$= 0$ ($\because x_n \to x$)

$\Rightarrow \operatorname*{Lt}_{n\to\infty} \|Tx_n - Tx\| = 0$ ($\because$ norm is always greater or equal to zero)

$\Rightarrow Tx_n \longrightarrow Tx.$

Hence T is continuous on X.

Conversely, suppose that T is continuous on X, we shall show that T is bounded. on contrary let us suppose that T is unbounded, then we can find a sequence $\{x_n\}$ in X such that:

$$\|Tx_n\| > n\|x_n\| \quad \forall n. \Rightarrow \frac{\|Tx_n\|}{n\|x_n\|} > 1 \quad \forall n.$$

let us choose $y_n = \frac{x_n}{n\|x_n\|}$, Then $y_n \in X$ as X is a linear space.

$$\Rightarrow T(y_n) = T\left(\frac{x_n}{n\|x_n\|}\right)$$

$\Rightarrow \|Ty_n\| = \left\|T\left(\frac{x_n}{n\|x_n\|}\right)\right\| = \frac{\|Tx_n\|}{n\|x_n\|} > 1$ (by above)

ie $\|Ty_n\| > 1.$

Since $y_n = \frac{x_n}{n\|x_n\|} \Rightarrow \|y_n\| = \left\|\frac{x_n}{n\|x_n\|}\right\| = \frac{\|x_n\|}{n\|x_n\|} = \frac{1}{n}$

$\Rightarrow \|y_n\| = \frac{1}{n} \to 0$ as $n \to \infty$

$\Rightarrow y_n \to 0$ as $n \to \infty$

$\Rightarrow Ty_n \to T(0) = 0$ ($\because$ T is continuous on X)

$\Rightarrow Ty_n \to 0 \Rightarrow \|Ty_n\| \to 0$

$\Rightarrow \|Tx_n\| \to 0$ $\left(\because \|Ty_n\| = \frac{\|Tx_n\|}{n\|x_n\|}\right)$

So $\|Ty_n\| = \frac{\|Tx_n\|}{n\|x_n\|} = 0 < 1$, which is a contridiction to the fact that $\|Ty_n\| > 1$, so our supposition was wrong and hence T is bounded. #.

Definition (1.14):
let X and Y be two normed linear spaces and let $T: X \longrightarrow Y$ be a bounded (Continuous) linear operator, then the norm of T is defined as:

$$\|T\| = \sup_{\|x\|=1} \|Tx\|$$

The norm of T is also defined by the following formulae.

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\| \quad \text{and} \quad \|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}.$$

Theorem (1.15): let $T: X \longrightarrow Y$ be a Continuous (bounded) linear operator from a n.l. space X into a n.l. space Y, Then

ⓐ $\|T\| < \infty$ ⓑ $\|Tx\| \leq \|T\|\|x\|$; $\forall x \in X$.

Proof: ⓐ since T is a bounded linear operator, so by definition, There exists a constant say $M>0$ such That $\|Tx\| \leq M\|x\|$; $\forall x \in X$

Then $\sup_{\|x\|=1} \|Tx\| \leq M \sup_{\|x\|=1} \|x\|$

$$\Rightarrow \sup_{\|x\|=1} \|Tx\| \leq M \cdot 1 \Rightarrow \sup_{\|x\|=1} \|Tx\| \leq M.$$

$$\Rightarrow \|T\| \leq M < \infty \quad (\text{by def. of } \|T\|)$$

$$\Rightarrow \|T\| < \infty.$$

ⓑ If $x=0$, Then the inequality is obvious.

If $x \neq 0$, Then put $y = \frac{x}{\|x\|}$ so That $\|y\| = 1$

Thus $Ty = T\left(\frac{x}{\|x\|}\right) = \frac{1}{\|x\|} \cdot Tx$. ($\because T$ is linear, $T(\alpha x) = \alpha Tx$)

$$\Rightarrow \|Ty\| = \frac{\|Tx\|}{\|x\|} \leq \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = \|T\| \quad \to (1).$$

$\|y\| = \left\|\frac{x}{\|x\|}\right\| = \frac{1}{\|x\|} \cdot \|x\| = 1$

Also $\|Ty\| = \dfrac{\|Tx\|}{\|x\|}$ gives

$$\|Tx\| = \|Ty\| \, \|x\| \leq \|T\| \, \|x\| \quad (\text{by } ①)$$

$$\Rightarrow \|Tx\| \leq \|T\| \, \|x\| \; ; \; \forall x \in X. \quad \# \text{proved}$$

Proposition: let T be a bounded linear operator. Then $x_n \to x \Rightarrow Tx_n \to Tx$.

Proof: Since $x_n \to x$. so that $\|x_n - x\| \to 0$ as $n \to \infty$ or $\lim\limits_{n\to\infty} \|x_n - x\| = 0$.

Now $\|Tx_n - Tx\| = \|T(x_n - x)\| \quad (\because T \text{ is linear})$

$\leq \|T\| \, \|x_n - x\| \quad (\because T \text{ is bounded})$

ie $\lim\limits_{n\to\infty} \|Tx_n - Tx\| = 0 \longrightarrow 0$ as $n \to \infty$

Hence $Tx_n \to Tx$ as $n \to \infty$.

Theorem (1.16): Suppose $T : X \to Y$ be a linear operator where X and Y are n.l.spaces. Then T^{-1} exists and is continuous on its domain of definition iff there exists a constant $m > 0$ such that:

$$m\|x\| \leq \|Tx\| \; ; \; \forall x \in X.$$

Proof:- suppose there exists a constant $m > 0$ such that $m\|x\| \leq \|Tx\| \; ; \; \forall x \in X \to ①$

In order to prove that T^{-1} exits, it is enough to show that $Tx = 0 \Rightarrow x = 0$. (Thm A).

suppose that $Tx = 0$, then ① becomes:

$$m\|x\| \leq \|0\| = 0 \Rightarrow m\|x\| = 0 \Rightarrow \|x\| = 0$$
$$\Rightarrow x = 0.$$

i.e. $Tx = 0$ implies $x = 0$

Thus T^{-1} exists.

Now To prove The Continuity of T^{-1}, we define

$Tx = y$, where $x \in X$ and $y \in Y$.

Since T^{-1} exists, so $T^{-1}y = x$.

Hence from ①, we have:

$m\|T^{-1}y\| \le \|y\| \Rightarrow \|T^{-1}y\| \le \frac{1}{m}\|y\|$

for all y in the range of T, which is the domain of T^{-1}.

~~So by Thm (1.12)~~

so that T^{-1} is bounded and by Thm (1.12) T^{-1} is continuous.

Conversely, if T^{-1} exists and is continuous, Then by Thm (1.12), T^{-1} is bounded and so we have:

$\|T^{-1}y\| \le \frac{1}{m}\|y\|$; $\forall$ y in The range of T.

i.e. $m\|T^{-1}y\| \le \|y\|$

But $Tx = y$ or $T^{-1}y = x$. so that

$m\|x\| \le \|Tx\|$; $\forall x \in X$.

which Completes The required Proof.

Definition (1.17):

① let X and Y be two normed linear spaces. A mapping T on X into Y is called an Isomorphism if the following conditions are satisfied.

ⓐ $T(x+y) = Tx + Ty$ ⓑ $T(\alpha x) = \alpha (Tx)$ $\forall x, y \in X$ and any scalar α.

ⓒ T is one-to-one.

ie if T is linear and one-to-one.

The spaces X and Y are said to be Isomorphic if there exists an Isomorphism of X onto Y (ie $T(X) = Y$).

② An Isometric Isomorphism between two normed linear spaces X and Y is an Isomorphism $T: X \longrightarrow Y$ such that

$\|Tx\| = \|x\|$; $\forall x \in X$.

The spaces X and Y are said to be Isometrically Isomorphic or Congruent if there exists an Isometric Isomorphism of X onto Y.

③ A topological Isomorphism between two normed linear spaces X and Y is an Isomorphism $T: X \longrightarrow Y$ such that T and T^{-1} are continuous in their respective domain.

The spaces X and Y are said to be topologically Isomorphic if there is a homeomorphism T of X onto Y that is also linear operator. For this reason X and Y may be called linearly homeomorphic.

Remark (1.18):

① In order to show that X and Y are congruent, it is necessary and sufficient that there exists a linear operator T with domain X and range Y such that $\|Tx\| = \|x\|$; $\forall x \in X$.

② Two norm linear spaces may be Isomorphic but not necessarly congruent. (Find example).

③ Topological Isomorphism is an equivalence relation ie it is reflexive, symmetric and Transitive.

Theorem (1.19): If X and Y are norm linear spaces they are topologically Isomorphic iff there exists a linear operator T with domain X and range Y and +ve constants m, M such that:

$$m\|x\| \le \|Tx\| \le M\|x\| \ ; \ \forall x \in X \longrightarrow ①$$

Proof: suppose that there exists a linear operator T with domain X and range Y and +ve constants m, M such that ① is satisfied.

we may write ① into two inequalities ie

$$m\|x\| \le \|Tx\| \ ; \ \forall x \in X \hookrightarrow ②$$

and $\|Tx\| \le M\|x\|$; $\forall x \in X$ $\hookrightarrow$ ③

Now by Thm (1.16) "T^{-1} exists and is continuous iff $m\|x\| \le \|Tx\|$; $\forall x \in X$ ie ② is satisfied".

Also by Thm (1.12), "T is continuous iff $\|Tx\| \le M\|x\|$ $\forall x \in X$ ie ③ is satisfied".

Hence combining the two results, we get:

T^{-1} exists and both T, T^{-1} are continuous if there exists constants $m>0$, $M>0$ such that

$$m\|x\| \le \|Tx\| \le M\|x\| \;;\; \forall x \in X.$$

which implies that X and Y are topologically isomorphic iff there exists a linear operator T with domain X and range Y and positive constants m & M such that:

$$m\|x\| \le \|Tx\| \le M\|x\| \;;\; \forall x \in X,$$

which completes the proof of the theorem.

T^{-1} exists
1) show bijective
condition: T, T^{-1}
2) show homeo: T

Definition (1.20):

let X be a linear space (or vector space). A norm $\|\cdot\|_1$ on X is said to be equivalent to a norm $\|\cdot\|_2$ on X iff there exists constants m, M both positive such that:

$$m\|x\|_1 \le \|x\|_2 \le M\|x\|_1 \;;\; \forall x \in X.$$

Theorem (1.21): let X be a linear space and suppose two norms $\|x\|_1$ and $\|x\|_2$ are defined on X. These norms define the same topology on X iff there exists +ve constants m, M such that

$$m\|x\|_1 \le \|x\|_2 \le M\|x\|_1 \;;\; \forall x \in X \quad (\text{ie they are equiv.})$$

Proof: let X_1, X_2 be the normed linear spaces that becomes with the norms $\|x\|_1$ and $\|x\|_2$ respectively.

ie $X_1 = (X, \|x\|_1)$, $X_2 = (X, \|x\|_2)$.

let us define $Tx = x$ and consider T as an operator with domain X_1 and range X_2 (ie $T: X_1 \to X_2$ is linear with domain $D(T) = X_1$ & range $R(T) = X_2$).

Suppose that there exists +ve constants m, M such that $m\|x\|_1 \le \|x\|_2 \le M\|x\|_1$; $\forall x \in X$.

Since $Tx = x$, so that

$$m\|x\|_1 \le \|Tx\|_2 \le M\|x\|_1 \; ; \forall x \in X.$$

Hence by Thm (1.19):

$m\|x\|_1 \le \|Tx\|_2 \le M\|x\|_1$; $\forall x \in X$ iff

X_1 and X_2 are topologically Isomorphic iff

T^{-1} exists and both T and T^{-1} are continuous iff

the open sets in X_1 are the same as the open sets in X_2 (by def: of continuity of $\underbrace{X_1 \,\&\, X_2}_{T \,\&\, T^{-1}}$).

Thus proving that the two norms define the same topology on X; since elements (open sets) of both the topologies are same.

which completes the required proof.

Finite dimensional norm linear spaces:

Theorem (1.22): Any two norm linear spaces of same finite dimension with the same scaler field are topologically Isomorphic.

Proof: let X_1, X_2 be two norm linear spaces of the same finite dimension with the same scaler field. we need to show that X_1 is topologically Isomorphic to X_2.
the case when $n=0$ is trivial. so we may assume that $n \geq 1$. It will suffice to prove that "if X is an n-dimensional n.l-space, it is topologically Isomorphic to $l'(n)$.
In order to prove that $l'(n)$ and X are topologically Isomorphic, we need to show that there exists a linear operator T with domain $l'(n)$ and range X and +ve constants m, M such that:

$$m\|\eta\| \leq \|T\eta\| \leq M\|\eta\| \quad ; \forall \eta \in l'(n)$$

(see Thm 1-20).

let $\{x_1, x_2, x_3, ---, x_n\}$ be a basis for X.
Define an operator $T: l'(n) \longrightarrow X$ by:

$$T(\eta) = \eta_1 x_1 + \eta_2 x_2 + \cdots + \eta_n x_n = \sum_{j=1}^{n} \eta_j x_j \quad \hookrightarrow (*)$$

for all $\eta = (\eta_1, \eta_2, ---, \eta_n) \in l'(n)$
then T is linear. we show that for all $\eta = (\eta_1, \eta_2, ---, \eta_n) \in l'(n)$, there exists $m>0$, $M>0$ such that

$$m\|\eta\| \leq \|T\eta\| \leq M\|\eta\|$$

that is

$$\|T\eta\| \leq M\|\eta\| \longrightarrow (1)$$

$$m\|\eta\| \leq \|T\eta\| \longrightarrow (2)$$

If $\eta = 0$, Then (1) and (2) are obviously true.

If $\eta \neq 0$, then by $(*)$,

$$\|T\eta\| = \|\sum_{j=1}^{n} \eta_j x_j\|$$

$$\leq \sum_{j=1}^{n} \|\eta_j x_j\|$$

$$= \sum_{j=1}^{n} |\eta_j| \|x_j\|$$

let us take $M = \max\{\|x_1\|, \|x_2\|, \ldots, \|x_n\|\}$

then $\|T\eta\| \leq M(|\eta_1| + |\eta_2| + \cdots + |\eta_n|)$

$$= M\|\eta\| \quad [\because \eta = (\eta_1, \eta_2, \ldots, \eta_n) \in l(n)]$$

which implies that (1) is true for $\eta \neq 0$.

From (2), note that:

$$m\|\eta\| \leq \|T\eta\| \text{ iff } m \leq \frac{\|T\eta\|}{\|\eta\|}$$

$$\text{iff } m \leq \frac{\|T(\eta_1, \eta_2, \ldots, \eta_n)\|}{\|\eta\|}$$

$$\text{iff } m \leq \|T(\beta)\|$$

where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, where

$$\beta_i = \frac{\eta_i}{\|\eta\|}, \quad \|\eta\| = |\eta_1| + |\eta_2| + \cdots + |\eta_n|$$

Then $\|\beta\| = 1$, because $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$

$$\Rightarrow \|\beta\| = \sum_{j=1}^{n} |\beta_j| = |\beta_1| + |\beta_2| + \cdots + |\beta_n|$$

$$= \frac{|\eta_1|}{\|\eta\|} + \frac{|\eta_2|}{\|\eta\|} + \cdots + \frac{|\eta_n|}{\|\eta\|}$$

$$= \frac{|\eta_1| + |\eta_2| + \cdots + |\eta_n|}{\|\eta\|}$$

$$= \frac{\|\eta\|}{\|\eta\|} = 1.$$

In order to prove ②, it is enough to show that there exists a constant $m > 0$ such that $m \leq \|T\beta\|$ for all $\beta \in l'(n)$ with $\|\beta\| = 1$.

we define a mapping $f : l'(n) \longrightarrow \mathbb{R}$ by

$f(\eta) = \|T\eta\|$ for all $\eta \in l'(n)$ $\longrightarrow$ (**)

then f is continuous function, because for any $\gamma \in l'(n)$, we have:

$$
\begin{aligned}
|f(\eta) - f(\gamma)| &= \big| \|T\eta\| - \|T\gamma\| \big| \\
&\leq \|T\eta - T\gamma\| \quad (\text{by Prop: } (1.5)) \\
&= \|T(\eta - \gamma)\| \quad (\because T \text{ is linear}) \\
&\leq c\,\|\eta - \gamma\| \quad , \text{where } c > 0
\end{aligned}
$$

ie $|f(\eta) - f(\gamma)| \leq c\,\|\eta - \gamma\|$, $c > 0$.

putting $\delta = \varepsilon/c$, we have:

$$\|\eta - \gamma\| < \delta \implies |f(\eta) - f(\gamma)| < \varepsilon$$

Thus f is continuous at $\gamma \in l'(n)$.

But γ is choosen arbitrary in $l'(n)$. Hence f is continuous on $l'(n)$.

Now we know (from Analysis) that "the surface of the unit sphere in $l^1(n)$ is compact", that is

$$K = \{\eta \in l^1(n) : \|\eta\| = 1\} \text{ is Compact in } l^1(n).$$

Hence the restriction of f to K namely g ie $f|_K = g$ is also Continuous, because f is Continuous. Also we know that "A real valued function on a compact set attains its maximum and minimum". Thus g attains its minimum, that is there exists $\eta \in K$ such that $g(\eta) \leq g(\beta)$; $\forall\ \beta \in K$, which yields $\|T\eta\| \leq \|T\beta\|$ (by ⊛⊛).

$$\Rightarrow \quad 0 \leq m \leq \|T\beta\| \text{ , where } \|T\eta\| = m$$

If $m = 0$, then $\|T\eta\| = 0$ iff $T\eta = 0$ iff $\sum_{j=1}^{n} \eta_j x_j = 0$

where $\eta = (\eta_1, \eta_2, ---, \eta_n)$.

But each η_i cannot be zero, because $\|\eta\| = 1$ (by def: of K).

So $\{x_1, x_2, \cdots, x_n\}$ is linearly dependent, which is a contradiction to the fact that $\{x_1, x_2, ---, x_n\}$ is linearly independent. Hence our supposition was wrong and so there exists a constant $m > 0$ such that $m \leq \|T\beta\|$; $\forall\ \beta \in l^1(n)$ with $\|\beta\| = 1$.

This implies that there exists a constant $m > 0$ such that $m\|\eta\| \leq \|T\eta\|$, which is the inequality (II).

So we have proved that there exists a linear operator T with domain $l^1(n)$ and range n-dimensional norm linear space X and constants $m>0$, $M>0$ such that:

$$m\|\eta\| \le \|T\eta\| \le M\|\eta\| \ ; \forall \eta \in l'(n).$$

Hence by Theorem (1.19), $l'(n)$ and X are topologically Isomorphic and Consequently X_1 and X_2 are topologically Isomorphic.

This Completes the required proof of the theorem.

<u>Remark:</u> let X and Y are topologically Isomorphic norm linear spaces and if one of them is complete (as a metric space), then other is also complete.

<u>Theorem (1.23):</u> A finite dimensional norm linear space is Complete.

<u>Proof:</u> By above remark, If X and Y are two topologically Isomorphic norm linear spaces, and if one of them is complete, then does the other.

Note that the space $l'(n)$ is topologically Isomorphic to the space $l'(1)$ (ie the real or complex field) which is Complete. Thus the finite dimensional norm linear space $l'(n)$ is Complete.

More generally, if X is any finite dimensional norm linear space, then we know that every finite dimensional norm linear space X is topologically Isomorphic to $l'(n)$ and hence X is complete. (by above Remark)

<u>Theorem (1.24)</u>: If X is a norm linear space, Then every finite dimensional subspace of X is necessarily closed.

<u>Proof</u>: let X be a norm linear space and M be a finite dimensional subspace of X, Then by above result, it is Complete.
Then by a result stating that " Every Complete subspace of a metric space is closed", we have That M is closed.

<u>Definition (1.25)</u>:
A metric space X is said to be Compact (or sequentially Compact) if every sequence in X has a Convergent subsequence.
A subset M of X is said to be Compact if every sequence in M has a Convergent subsequence whose limit is an element of M.

<u>Theorem (1.26)</u> <u>Continuous mapping Theorem</u>
let X and Y be metric spaces and $T: X \to Y$ be Continuous mapping, then the image of a Complete Subset M of X under T is Compact.

Theorem (1.27): If X is a finite dimensional normed linear space then each closed and bounded set in X is compact.

Proof: Let X be a finite dimensional norm linear space and M be a closed and bounded set in X. We show that M is compact in X.

We know that "Any two norm linear spaces of the same finite dimension with the same scaler field are topologically Isomorphic", so there exists a topological Isomorphism $T: X \xrightarrow{\text{onto}} l^1(n)$.

Since $M \subset X$, then $T(M) = K$, closed and bounded in $l^1(n)$. [$\because T$ is a homeomorphism], and so K is compact. [using Hiene-Borel theorem], because in space $l^1(n)$, we have from analysis that "each closed and bounded set in $l^1(n)$ is always compact".

Since T^{-1} exits (ie $T^{-1}: l^1(n) \to X$) and is continuous so using the fact that "Continuous image of compact set is compact", we can say that $T^{-1}(K) = M$ is compact in X. which completes the proof.

Theorem (1-28): If X is finite dimensional n.l.space, Then each compact subset M of X is closed and bounded.

Proof: we know that "each compact set in a metric space is closed and bounded". Hence if M is a compact set in X, it must be closed and bounded. ($\because$ every n.l. space is a metric space)

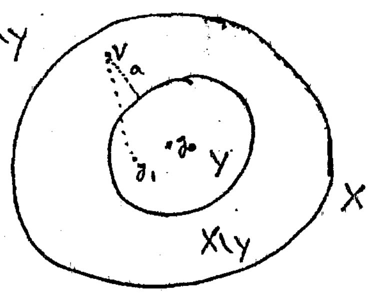
Remark: Combining Thm (1.27) and Thm (1.28), we have the following theorem.

Theorem (1.29): If X is a finite dimensional norm linear space, Then each subset of X is Compact iff it is closed and bounded.

(1.30) Lemma (F. Riesz's Lemma):

let Y be a Subspace of a norm linear space X (of any dimension) such That Y is closed and a proper subset of X, then for every real number θ in the interval $(0,1)$, There exists a vector $x \in X$ such That $\|x\| = 1$ and $\|x-y\| \geq \theta \quad \forall y \in Y$.

Proof: we Consider any vector $v \in X \setminus Y$ and denote its distance from Y by a, That is



$$a = \inf_{y \in Y} \|v-y\|$$

clearly $a > 0$ [because norm is always non-negative but $v \in X \setminus Y$]

Since Y is closed. we Know $\theta \in (0,1)$. By definition of an infimum, There is a $y_0 \in Y$ such That

$$a \leq \|v-y_0\| \leq \frac{a}{\theta} \quad \left(\text{since } \theta \in (0,1), \text{ so } a < \frac{a}{\theta}\right)$$

let $x = c(v-y_0)$, where $c = \dfrac{1}{\|v-y_0\|}$

Then $\|x\| = \|c(v-y_0)\| = c\|v-y_0\| = \dfrac{1}{\|v-y_0\|} \cdot \|v-y_0\|$

ie $\|x\| = 1$.

and we remain to show that $\|x-y\| \geq \theta \ \forall y \in Y$.

Now $\|x-y\| = \|c(v-y_0)-y\| = c\,\|(v-y_0) - c^{-1}y\|$

$$= c\,\|v-(y_0+c^{-1}y)\|$$

$$= c\,\|v-y_1\|, \text{ where } y_1 = y_0 + c^{-1}y.$$

The form of y_1 shows that $y_1 \in Y$ ($\because Y$ is a subspace)

Hence $\|v-y_1\| \geq a$ $\left(\because a = \inf\limits_{y\in Y} \|v-y\|, \text{ by graph}\right)$.

Now
$$\|x-y\| = c\,\|v-y_1\|$$
$$\geq c.a$$
$$= \frac{1}{\|v-y_0\|}.a \qquad \left(\because c = \frac{1}{\|v-y_0\|}\right)$$
$$\geq \frac{a}{a/\theta} \qquad \left(\because a \leq \|v-y_0\| \leq \frac{a}{\theta}\right)$$
$$= \theta$$

So that $\|x-y\| \geq \theta$, where $\theta \in (0,1)$.

Since $y \in Y$ was choosen arbitrary; Therefore $\|x-y\| \geq \theta$; $\forall\, y \in Y$. This completes the proof.

Theorem (1.31) (Converse of 1.27)

let X be a norm linear space and suppose that the surface of the unit sphere $S = \{x \in X : \|x\| = 1\}$ in X is compact, then X is finite dimensional.

Proof: let X be a norm linear space. We need to show that X is finite dimensional.

We assume that $\dim X = \infty$, But S is compact in X, and we show that this leads to a contradiction.

we choose any $x_1 \in S$. Define $X_1 = \langle x_1 \rangle$
ie x_1 generates a one dimensional space X_1 of X. Then X_1 is closed (by Thm 1.124) and is a proper subspace of X, because $\dim X = \infty$.

Hence by Riez's Lemma, There is $x_2 \in S$ such that $\|x_2 - x_1\| \geq \theta = \frac{1}{2}$

Define $X_2 = \langle x_1, x_2 \rangle$, a two dimensional space generated by x_1, x_2 in S. So X_2 is a proper closed subspace of X. Again by Riez's Lemma, There is an $x_3 \in S$ such that for all $x \in X$, we have:

$$\|x_3 - x\| \geq \tfrac{1}{2}.$$

In particular, if $x = x_1$, then $\|x_3 - x_1\| \geq \tfrac{1}{2}$.

and if $x = x_2$, then $\|x_3 - x_2\| \geq \tfrac{1}{2}$.

Proceeding by induction, we obtain an infinite sequence $\{x_n\}$ in S such that

$$\|x_m - x_n\| \geq \tfrac{1}{2} \quad (m \neq n).$$

obviously $\{x_n\}$ Cannot have a convergent subsequence because $\{x_n\}$ itself ~~Cannot~~ is not a convergent sequence. This facts Contradicts the Compactness of S ($\because S$ is compact iff every sequence in S converges to a point in S)
Hence our supposition that $\dim X = \infty$ was false, and so $\dim X < \infty$.

— * — * — * — * — * — * — * —

Theorem (1.32): on a finite dimensional norm linear space, any two norms are equivalent.

Proof: Before proving this result, we state the following lemma.

" If $\{x_1, x_2, ---, x_n\}$ be a linearly independent set in n.l. space, Then there exists a constant $c>0$ such That for each scalers $\alpha_1, \alpha_2, ----, \alpha_n$

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \geq c \sum_{i=1}^{n} |\alpha_i| \text{"}.$$

Now we prove the required result.

let $\{x_1, x_2, ---, x_n\}$ be a basis for X.

If $x \in X$, so it can be uniquely expressed as

$$x = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n = \sum_{i=1}^{n} \xi_i x_i . \quad \hookrightarrow (1)$$

where ξ_i are scalers.

let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms defined on X.

By above lemma, There exists a constant $c>0$ such That:

$$\|x\|_1 = \left\| \sum_{i=1}^{n} \xi_i x_i \right\|_1 \geq c \sum_{i=1}^{n} |\xi_i| \hookrightarrow (2)$$

Since $\|\cdot\|_2$ is a norm on X, so

$$\|x\|_2 = \left\| \sum_{i=1}^{n} \xi_i x_i \right\|_2$$

$$\leq \sum_{i=1}^{n} \| \xi_i x_i \|_2 \qquad (\because \|x_1+x_2\| \leq \|x_1\| + \|x_2\|)$$

$$= \sum_{i=1}^{n} |\xi_i| \|x_i\|_2$$

$$\leq K \sum_{i=1}^{n} |\xi_i|$$

expand

where $K = \max \|x_i\|_2$, $1 \le i \le n$.

$c > 0$
$K = \max \|x_i\|_2$
so $K > 0$
$\therefore c/K > 0$.

So that $\frac{c}{K}\|x\|_2 \le \frac{c}{K} \cdot K \sum_{i=1}^{n} |\xi_i|$

$$\Rightarrow m\|x\|_2 \le c \sum_{i=1}^{n} |\xi_i| \text{ , where } m = c/K.$$

$\therefore m > 0$

$$\Rightarrow m\|x\|_2 \le \|x\|_1 \longrightarrow (3)$$

If we interchange the role of $\|\cdot\|_1$ and $\|\cdot\|_2$, we get:

$$\|x\|_1 \le M\|x\|_2 \text{ , where } M > 0. \longrightarrow (4)$$

Combining (3) and (4), we get:

$$m\|x\|_2 \le \|x\|_1 \le M\|x\|_2 \quad \text{where } M > 0,\ m > 0.$$

Hence $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

__Proposition (1.33)__: If $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, p and q are Holder Conjugate (or simply Conjugate) of each other, then for $a \ge 0$, $b \ge 0$, we have the following inequality $a^{1/p} \cdot b^{1/q} \le \frac{a}{p} + \frac{b}{q}$.

__Proof:__ If $a = 0$ or $b = 0$, proposition is clearly satisfied.

we assume the case when both $a > 0$, $b > 0$.

Now if $k \in (0,1)$, define $f(t)$ for $t \ge 1$ by:

$$f(t) = k(t-1) - t^k + 1. \longrightarrow (1)$$

Note that $f(1) = 0$ and $f(t) \ge 0$ for all other values of t.

we have $0 \le f(t) = k(t-1) - t^k + 1$

$$\Rightarrow t^k \le kt + (1-k) \longrightarrow (2)$$

If $a \geq b$, then put $t = \frac{a}{b}$ and $k = \frac{1}{p}$.

so that ② becomes:

$$\left(\frac{a}{b}\right)^{1/p} \leq \frac{1}{p}\left(\frac{a}{b}\right) + \left(1 - \frac{1}{p}\right)$$

$$\Rightarrow a^{1/p} \cdot b^{-1/p} \leq \frac{1}{p} \cdot \frac{a}{b} + \frac{1}{q} \quad \left(\because \frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{1}{q} = 1 - \frac{1}{p}\right)$$

$$\Rightarrow a^{1/p} \cdot b^{1-1/p} \leq \frac{a}{p} + \frac{b}{q} \quad (\text{mult: by } b)$$

$$\Rightarrow a^{1/p} \cdot b^{1/q} \leq \frac{a}{p} + \frac{b}{q}.$$

Now if $a < b$, then put $t = b/a$, $k = 1/q$

so that ② becomes:

$$\left(\frac{b}{a}\right)^{1/q} \leq \frac{1}{q} \cdot \frac{b}{a} + \left(1 - 1/q\right)$$

$$\Rightarrow a^{-1/q} \cdot b^{1/q} \leq \frac{1}{q} \cdot \frac{b}{a} + \frac{1}{p} \left(\because \frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{1}{p} = 1 - 1/q\right)$$

$$\Rightarrow a^{1-\frac{1}{q}} \cdot b^{1/q} \leq \frac{b}{q} + \frac{a}{p} \quad (\text{mult: by } a)$$

$$\Rightarrow a^{1/p} \cdot b^{1/q} \leq \frac{a}{p} + \frac{b}{q}.$$

which completes the proof.

(1.34) Hölder Inequality:

If $1 < p < \infty$ and $x = (x_1, x_2, \dots, x_n)$,

$y = (y_1, y_2, \dots, y_n)$; Then

$$\sum_{i=1}^{n} |x_i y_i| \le \|x\|_p \, \|y\|_q \quad , \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

Proof: we know that for $x = (x_1, x_2, \dots, x_n)$,

$y = (y_1, y_2, \dots, y_n)$.

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \text{ and } \|y\|_q = \left(\sum_{i=1}^{n} |y_i|^q \right)^{1/q}$$

If $x = 0$ or $y = 0$, Then the inequality is obvious.

we assume that x, y are both non-zero, Then

assume that:

$$a_i = \left(\frac{|x_i|}{\|x\|_p} \right)^p \quad , \quad b_i = \left(\frac{|y_i|}{\|y\|_q} \right)^q \longrightarrow (1)$$

Then by proposition (1.33), we have:

$$a_i^{1/p} \cdot b_i^{1/q} \le \frac{a_i}{p} + \frac{b_i}{q} \longrightarrow (2)$$

Therefore
$$\frac{|x_i y_i|}{\|x\|_p \|y\|_q} = \frac{|x_i|}{\|x\|_p} \cdot \frac{|y_i|}{\|y\|_q}$$

$$= a_i^{1/p} \cdot b_i^{1/q} \quad [\text{by } (1)]$$

$$\le \frac{a_i}{p} + \frac{b_i}{q} \quad [\text{by } (2)].$$

$$= \frac{\left(\frac{|x_i|}{\|x\|_p} \right)^p}{p} + \frac{\left(\frac{|y_i|}{\|y\|_q} \right)^q}{q} \quad [\text{by } (1)]$$

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ie $$\frac{|x_i y_i|}{\|x\|_p \|y\|_q} \leq \frac{\left(\frac{|x_i|}{\|x\|_p}\right)^p}{p} + \frac{\left(\frac{|y_i|}{\|y\|_q}\right)^q}{q}.$$

Taking finite summation of both sides,

$$\sum_{i=1}^{n} \frac{|x_i y_i|}{\|x\|_p \|y\|_q} \leq \sum_{i=1}^{n} \frac{\left(\frac{|x_i|}{\|x\|_p}\right)^p}{p} + \sum_{i=1}^{n} \frac{\left(\frac{|y_i|}{\|y\|_q}\right)^q}{q}$$

$$= \sum_{i=1}^{n} \frac{\frac{|x_i|^p}{\|x\|_p^p}}{p} + \sum_{i=1}^{n} \frac{\frac{|y_i|^q}{\|y\|_q^q}}{q}.$$

$\hookrightarrow$ ③

Thus by above recall and ③, we have:

$$\sum_{i=1}^{n} \frac{|x_i y_i|}{\|x\|_p \|y\|_q} \leq \frac{\frac{\|x\|_p^p}{\|x\|_p^p}}{p} + \frac{\frac{\|y\|_q^q}{\|y\|_q^q}}{q}$$

$$= \frac{1}{p} + \frac{1}{q}.$$

$$= 1 \quad \text{(Given)}$$

$$\Rightarrow \sum_{i=1}^{n} |x_i y_i| \leq \|x\|_p \|y\|_q \quad \text{for } x = (x_1, x_2, \ldots, x_n)$$
$$y = (y_1, y_2, \ldots, y_n)$$

which completes the proof.

Remark: when $p = q = 2$; we have:

$\sum_{i=1}^{n} |x_i y_i| \leq \|x\|_2 \|y\|_2$, which is called the Cauchy's inequality.

(1.35) Minkowski's Inequality:

If $1 \le p < \infty$ and $x = (x_1, x_2, \ldots, x_n)$,

$y = (y_1, y_2, \ldots, y_n)$, then $\|x+y\|_p \le \|x\|_p + \|y\|_p$.

Proof: For $p=1$, the inequality is simply the triangle inequality. So we assume that $1 < p < \infty$.

we have $x+y = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$

thus $\|x+y\|_p = \left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{1/p}$

So
$$\|x+y\|_p^p = \sum_{i=1}^{n} |x_i + y_i|^p$$

$$= \sum_{i=1}^{n} |x_i+y_i| \, |x_i+y_i|^{p-1}$$

$$\le \sum_{i=1}^{n} \left(|x_i| + |y_i| \right) |x_i + y_i|^{p-1}$$

$$= \sum_{i=1}^{n} |x_i| \, |x_i+y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \, |x_i+y_i|^{p-1}$$

$$\le \|x\|_p \, \|x+y\|_p^{p-1} + \|y\|_p \, \|x+y\|_p^{p-1}$$

(By applying Holder inequality)

$$= \left(\|x\|_p + \|y\|_p \right) \|x+y\|_p^{p-1}.$$

ie $\|x+y\|_p^p \le \left(\|x\|_p + \|y\|_p \right) \|x+y\|_p^{p-1}$.

$\Rightarrow \|x+y\|_p^{p-(p-1)} \le \|x\|_p + \|y\|_p$ (Dividing by $\|x+y\|_p^{p-1}$)

$\Rightarrow \|x+y\|_p \le \|x\|_p + \|y\|_p$

which is the desired Minkowski's inequality.

Remark: For $P = 2$, we have from above:

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$$

which is the famous Schwarz's inequality.

The end chapter # 1

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CHAPTER # 2 { BANACH SPACES }

Definition (2.1): ⓐ A normed linear space X is called complete if every Cauchy sequence in X converges to a limit point in it.

ⓑ A complete normed linear space is called a Banach space. OR

A normed linear space which is complete as a metric space is called a Banach space.

Remark (2.2): If a normed linear space is not complete, we may complete it by adjoining certain ideal elements to X so as to obtain a complete space $\hat{X}$.

ie X may be inlarged to form a Banach space $\hat{X}$ in which X is dense (ie $X \subset \hat{X}$, $\overline{X} = \hat{X}$).

The complete space $\hat{X}$ is called the "Banach space completion of X".

The space $\hat{X}$ is essentially unique, in the sense that any other Banach space containing X as a dense subspace must be isometrically isomorphic to $\hat{X}$.

Theorem (2.3): Let X and Y be normed linear spaces and let T be a continuous linear operator on X into Y. Then there exists a uniquely determined continuous linear operator $\hat{T}$ of $\hat{X}$ into $\hat{Y}$ such that $\hat{T}x = Tx$ if $x \in X$

and $\|\hat{T}\| = \|T\|$.

Proof: In order to obtain $\hat{T}$, we suppose that $\hat{x} \in \hat{X}$ and select a sequence $\{x_n\}$ in X such that $x_n \longrightarrow \hat{x}$. Then $\{x_n\}$ is a Cauchy sequence ($\because$ every cgt seq: is a Cauchy sequence).

Now $\|Tx_n - Tx_m\| = \|T(x_n - x_m)\|$ ($\because T$ is linear)

$\leq \|T\| \|x_n - x_m\|$ ($\because T$ is bounded)

$\longrightarrow 0$ as $m, n \longrightarrow \infty$ ($\because \{x_n\}$ is a Cauchy sequence)

Therefore $\{Tx_n\}$ is a Cauchy sequence in Y and hence in $\hat{Y}$. So $\{Tx_n\}$ is convergent in $\hat{Y}$ because $\hat{Y}$ is complete.

Let $Tx_n \longrightarrow \hat{y} \in \hat{Y}$.

Now we define $\lim_{n \to \infty} Tx_n = \hat{T}\hat{x} \longrightarrow (1)$

Then $\hat{T}$ is a map on $\hat{X}$ into $\hat{Y}$

Now $\hat{T}$ is linear, for if $\hat{T}\hat{x}_1 = \lim_{n \to \infty} Tx_n$ (by (1))

and $\hat{T}\hat{x}_2 = \lim_{n \to \infty} Ty_n$, Then

$$\hat{T}(\hat{x_1}+\hat{x_2}) = \lim_{n\to\infty} T(x_n+y_n)$$

$$= \lim_{n\to\infty} Tx_n + \lim_{n\to\infty} Ty_n \quad (\because T \text{ is linear})$$

$$= \hat{T}\hat{x_1} + \hat{T}\hat{x_2}. \quad (\text{by } 1)$$

and $\hat{T}(\alpha\hat{x}) = \lim_{n\to\infty} T(\alpha x_n)$ [by 1].

$$= \alpha \lim_{n\to\infty} Tx_n \quad [\because T \text{ is linear}]$$

$$= \alpha \hat{T}\hat{x}.$$

Hence $\hat{T}$ is linear.

Next we show that $\hat{T}$ is continuous.

If $x \in X$, then $x \in \hat{X}$ and so by 1, we have

$$\hat{T}x = \lim_{n\to\infty} Tx_n = T\lim_{n\to\infty} x_n \quad [\because T \text{ is } \cancel{\text{linear}} \text{ continuous}]$$

$$= Tx \quad [\because x_n \to \hat{x}].$$

Hence $\hat{T}x = Tx$ if $x \in X \longrightarrow (2)$

Also since $x_n \to \hat{x}$, so $Tx_n \to T\hat{x}$ [$\because T$ is continuous].

ie $\lim_{n\to\infty} Tx_n = T\hat{x} \longrightarrow (3)$

and hence $\|\hat{T}\hat{x}\| = \|\lim_{n\to\infty} Tx_n\|$ (by 1)

$$= \|T\hat{x}\| \quad (\text{by } 3).$$

$$\leq \|T\|\|\hat{x}\| \quad [\because \|Tx\| \leq \|T\|\|x\| \ \forall x \in X]$$

which shows that $\hat{T}$ is bounded and hence continuous.

Also $\|\hat{T}\| \leq \|T\|$ (by def: of norm of an operator). $\to$ (4)

we need to show that $\|T\| \leq \|\hat{T}\|$

we have by (2), $\|Tx\| = \|\hat{T}x\|$

$\leq \|\hat{T}\| \|x\|$ [$\because \hat{T}$ is linear]

So that

$\|T\| \leq \|\hat{T}\|$ (by def: of norm of an operator) $\to$ (5)

Form (4) & (5), we get:

$\|\hat{T}\| = \|T\|$.

Since X is considered to be dense in $\hat{X}$, so $\hat{T}$ thus obtained is unique.

this completes the proof.

<u>Examples:</u> (1) The spaces $\mathbb{R}$ and $\mathbb{C}$ of real and complex numbers are Banach spaces.

(2) The space $\mathbb{R}^n$ and $\mathbb{C}^n$ are Banach spaces.

<u>Sol</u>: (1) The spaces $\mathbb{R}$ and $\mathbb{C}$ with norm defined by $\|x\| = |x|$ are normedlinear spaces (by chap 1).

we also know from "Analysis" that every Cauchy sequence of real and complex numbers converges ie $\mathbb{R}$ and $\mathbb{C}$ are complete. Thus $\mathbb{R}$ and $\mathbb{C}$ are complete normedlinear spaces and hence Banach spaces.

Recall: Let X be a Complete metric space &

(a) Y be a complete subspace of X, Then Y is closed.

(b) Y be a closed subspace of X, Then Y is Complete.

Theorem (2.4): A linear subspace X_0 of a Banach space X is itself a Banach space iff X_0 is closed.

Proof: Let X_0 is a Complete n.l. subspace of The Banach space X. Since X is a Banach space, so it is Complete as a metric space ($\because$ every n.l. space is a metric space).

Since X_0 is a Complete subspace of The Complete space X, so by above recall (a), X_0 is closed.

Conversely, let X_0 be a closed subspace of The Complete space X; Then by reall (b), X_0 is complete. ie X_0 is a Banach space.

which completes the required proof.

Definition (2.5): Let X be a normedlinear space and let $x_1+x_2+\cdots+x_n+\cdots = \sum_{i=1}^{\infty} x_i$ be a formal series of elements of X and $S_n = x_1+x_2+\cdots+x_n = \sum_{i=1}^{n} x_i$ be the partial sum of elements of X, Then $\{S_n\}$ is a sequence in X.

(a) we say that the series $\sum_{i=1}^{\infty} x_i$ is Convergent (or summable) in X if there is an element $x \in X$ such that The sequence $\{S_n\}$ of its partial sum Converges to x and we write $x = \sum_{i=1}^{\infty} x_i$.

(b) we say that the series $\sum_{i=1}^{\infty} x_i$ is absolutely Convergent (or absolutely summable) if $\sum_{i=1}^{\infty} \|x_i\|$ is Convergent ie $\sum_{i=1}^{\infty} \|x_i\| < \infty$.

Theorem (2.6): A normedlinear space X is a Banach space iff every absolutely Convergent series in X is Convergent.

Proof: Let X be a Banach space. let $\sum_{i=1}^{\infty} x_i$ be any absolutely convergent series in X ie $\sum_{i=1}^{\infty} \|x_i\|$ is Convergent ie $\sum_{i=1}^{\infty} \|x_i\| < \infty$. we shall show that $\sum_{i=1}^{\infty} x_i$ is Convergent.

Since $\sum_{i=1}^{\infty} \|x_i\|$ is convergent; So by definition of Convergent series, the sequence of its partial sums $\{t_n\}$ is Convergent, where $t_n = \sum_{i=1}^{n} \|x_i\|$. Thus $\{t_n\}$ must be a Cauchy sequence ($\because$ every cgt sequence is Cauchy seq.)

let $\{S_n\}$ be the sequence of partial sums of the series $\sum_{i=1}^{\infty} x_i$, where $S_n = \sum_{i=1}^{n} x_i$.

Now for $m > n$, we have:

$$\|S_m - S_n\| = \|x_1 + x_2 + \cdots + x_n + x_{n+1} + \cdots + x_m - x_1 - x_2 - \cdots - x_n\|$$

$$= \|x_{n+1} + x_{n+2} + \cdots + x_m\|$$

$$= \left\|\sum_{i=n+1}^{m} x_i\right\| \le \sum_{i=n+1}^{m} \|x_i\| = \|t_m - t_n\| \quad \left(\because t_n = \sum_{i=1}^{n} \|x_i\|\right)$$

$$\longrightarrow 0 \text{ as } m, n \to \infty \ (\because t_n \text{ is a Cauchy sequence})$$

$\|t_m - t_n\| = \|\,\|x_1\| + \|x_2\| + \cdots + \|x_m\| - \|x_1\| - \cdots - \|x_n\|\,\| = \|\,\|x_{n+1}\| + \|x_{n+2}\| + \cdots + \|x_m\|\,\|$

ie $\|s_m - s_n\| \to 0$ as $m, n \to \infty$

ie $\{s_n\}$ is a Cauchy sequence in X. But X is Complete ($\because X$ is a Banach space), so $\{s_n\}$ is Convergent in X. Thus by definition, the series $\sum_{i=1}^{\infty} x_i$ is Convergent in X (because the seq: of its partial sum cgs).

Conversely, let X be a normed linear space and let every absolutely convergent series in X is Convergent. we have to show that X is a Banach space ie X is Complete. For this let $\{x_n\}$ be a Cauchy sequence in the n.l. space X; then by induction, it is possible to select a subsequence say $\{u_K\}$ of $\{x_n\}$ such that:

$$\|u_{K+1} - u_K\| < 2^{-K}; \quad K = 1, 2, \ldots$$

$$\Rightarrow \sum_{K=1}^{\infty} \|u_{K+1} - u_K\| < \sum_{K=1}^{\infty} 2^{-K} < \infty$$

$\sum_{K=1}^{\infty} 2^{-K}$, $K=1,2,\ldots$
$= 2^{-1} + 2^{-2} + 2^{-3} + \cdots$
$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$
$= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)$
$= \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\frac{1}{2}} = 1$

ie $\sum_{K=1}^{\infty} \|u_{K+1} - u_K\| < \infty$

$\Rightarrow$ The series $\sum_{K=1}^{\infty} (u_{K+1} - u_K)$ is absolutely Convergent.

Since every absolutely Convergent series in X is Convergent, so it follows that $\sum_{K=1}^{\infty} (u_{K+1} - u_K)$ is convergent. So by definition, the sequence of its partial sums $\{y_n\}$ is Convergent in X, where

$$y_n = \sum_{K=1}^{n} (u_{K+1} - u_K) = u_{n+1} - u_1 \quad \text{[After expanding]}$$

This gives that the subsequence $\{u_K\}$ converges to some $x \in X$.

We see that the Cauchy sequence $\{x_n\}$ has a convergent subsequence $\{u_K\}$ and therefore the whole sequence $\{x_n\}$ converges ($\because$ if a Cauchy sequence has a cgt subsequence, then the whole seq. is convergent). Consequently X is a complete norm linear space ie X is a Banach space. which completes the proof.

Assignment (2.7): Show that the space l^p, where $p \geq 1$ is complete ie a Banach space.

Proof: The space l^p, where $p \geq 1$ is defined to consist of all sequences $x = \{\xi_n\}$ such that $\sum_{n=1}^{\infty} |\xi_n| < \infty$. The norm in l^p is defined by:

$$\|x\| = \left(\sum_{n=1}^{\infty} |\xi_n|^p \right)^{1/p}$$

Then with this norm, l^p, $p \geq 1$ is a normed linear space. Thus to show that l^p space is a Banach space, it only remains to show that it is complete. Let $\{x_n\}$ be a Cauchy sequence in l^p with

$x_n = (\xi_1^{(n)}, \xi_2^{(n)}, \ldots\ldots)$. For each K, $\{\xi_K^{(n)}\}$ is a Cauchy sequence, because

$$\left| \xi_K^{(n)} - \xi_K^{(m)} \right| \leq \left(\sum_{i=1}^{\infty} \left| \xi_i^{(n)} - \xi_i^{(m)} \right|^p \right)^{1/p} = \|x_n - x_m\| \longrightarrow 0 \text{ as } m,n \to \infty$$

Let us suppose that $\lim_{n \to \infty} \xi_K^{(n)} = \xi_K^{(m)} \quad \text{--- (1)}$

we shall show that the sequence $\{\xi_K\}$ is an element of l^p. We know that $\{x_n\}$ is bounded, so $\|x_n\| \leq M$ ($\because$ Every Cauchy sequence is bounded).

Now for any K

$$\left(\sum_{i=1}^{K} |\xi_i^{(n)}|^p\right)^{1/p} \leq \|x_n\| \leq M.$$

Letting $n \to \infty$, we obtain

$$\left(\sum_{i=1}^{K} |\xi_i|^p\right)^{1/p} \leq M \quad (\text{by } ①).$$

Since K is arbitrary, it follows that $\{\xi_K\}$ is an element of l^p and that its norm does not exceed M. Let $x = \{\xi_K\}$. It remains to prove that $\|x_n - x\| \to 0$ as $n \to \infty$.

Suppose $\varepsilon > 0$, then there exists an integer N such that

$\|x_n - x_m\| < \varepsilon$ if $m, n \geq N$ ($\because$ $\{x_n\}$ is a Cauchy seq.)

therefore for any K,

$$\left(\sum_{i=1}^{K} |\xi_i^{(n)} - \xi_i^{(m)}|^p\right)^{1/p} \leq \|x_n - x_m\|$$

$$< \varepsilon \text{ if } m, n \geq N.$$

Keeping K and n fixed, let $m \to \infty$, we then get

$$\left(\sum_{i=1}^{K} |\xi_i^{(n)} - \xi_i|^p\right)^{1/p} < \varepsilon \text{ if } n \geq N$$

This is true for all K, we can let $K \to \infty$ and obtain the result that $\|x_n - x\| < \varepsilon$ if $n \geq N$

This shows that the Cauchy sequence is convergent in l^p. So that l^p is complete. Hence l^p, $p \geq 1$ is a Banach space.

Dual spaces:

Definition (2.8): Let X be a n.l.s and let $\mathbb{K} (=\mathbb{R}$ or $\mathbb{C})$ be a scaler field associated with X. This field is also a normed linear space with norm defined as: $\|x\| = |x|$; $x \in \mathbb{K}$, Then

① A linear operator $x' : X \longrightarrow \mathbb{K}$ (scaler field assoc: with X) is called functional.

② A functional $x' : X \longrightarrow \mathbb{K}$ is said to be Continuous at a point x_0 of X if for each $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\|x - x_0\| < \delta \Rightarrow |x'(x) - x'(x_0)| < \varepsilon.$$

we say that x' is Continuous on X iff it is continuous at each point of X.

③ A functional $x' : X \longrightarrow \mathbb{K}$ is said to be linear if

(i) $x'(x_1 + x_2) = x'x_1 + x'x_2$.

(ii) $x'(\alpha x) = \alpha x'x$, where $x, x_1, x_2 \in X$ and α is any scaler.

④ A linear functional x' is said to be bounded if there exists a constant $M > 0$ such that

$$|x'(x)| \le M \|x\| \; ; \; \forall x \in X.$$

$x'(x)$ is a number (real or complex). So we take absolute in place of norm

⑤ The set of all linear functionals defined on X is itself a linear space, if addition and scaler multiplication are defined by:

$$(x_1' + x_2')(x) = x_1'(x) + x_2'(x).$$

$$(\alpha x')(x) = \alpha x'(x)$$

and is denoted by X^f, called the Algebraic dual (or Conjugate) space of X.

(6) A norm of a linear functional $x' \in X^f$ is defined as:

$$\|x'\| = \sup_{\|x\|=1} |x'x|$$

(As in case of T, $\|T\| = \sup_{\|x\|=1} \|Tx\|$)

$$= \sup_{\|x\| \leq 1} |x'x|$$

$$= \sup_{x \neq 0} \frac{|x'x|}{\|x\|}$$

Note that $|x'x| \leq \|x'\| \|x\|$; $\forall x \in X$.

(7) The set of all bounded (continuous) linear functionals defined on X is a linear subspace of X^f and is denoted by X'.

A norm on X' is given by (6). The linear space X' normed in this way is called normed conjugate of X. Sometimes it is denoted by X^*.

<u>Remark</u>: Since X^f is a linear space, we may also consider its algebraic dual (or conjugate) space, which we denote by $(X^f)^f$ or X^{ff}. That is the class of all linear functionals on X^f. We shall denote elements of X^{ff} by x'' (i.e. $x'' : X^f \longrightarrow \mathbb{K}$, the scalar field associated with X^f) and we shall use the notation $x''(x')$ for the value of x'' at x'.

Theorem (2.9): Let X be a norm linear space. Then the norm conjugate space X' of X is complete.

Proof: Let $\{x'_n\}$ be a Cauchy sequence in X', then by definition of Cauchy sequence; for every $\varepsilon > 0$, there exists +ve integer N such that

$\|x'_m - x'_n\| < \varepsilon$ whenever $m, n \geq N$ ⟶ ①

Consequently for each $x \in X$

$$|x'_m(x) - x'_n(x)| = |(x'_m - x'_n)x| \leq \|x'_m - x'_n\| \|x\|$$
$$< \varepsilon \|x\|, \ \forall \ m, n \geq N \longrightarrow ②$$

which shows that $\{x'_n(x)\}$ is a Cauchy sequence in the space $\mathbb{R}$ or $\mathbb{C}$ ($\because x' : X \to K (= \mathbb{R}$ or $\mathbb{C})$) for each $x \in X$. Since the scaler field $\mathbb{R}$ or $\mathbb{C}$ are complete, so $\{x'_n(x)\}$ converges to a limit depending on X which we denote by $x'(x)$.

i.e. $\lim\limits_{n \to \infty} x'_n(x) = x'(x)$ ⟶ ③

Thus defining a functional x' on X. We show that $x' \in X'$ and for this it is enough to show that x' is linear and bounded.

x' is linear: Since for scalers λ_1, λ_2 and vectors x_1, x_2 in X, we have:

$$x'(\lambda_1 x_1 + \lambda_2 x_2) = \lim_{n \to \infty} x'_n(\lambda_1 x_1 + \lambda_2 x_2) \quad [\text{using ③}].$$
$$= \lim_{n \to \infty} x'_n(\lambda_1 x_1) + \lim_{n \to \infty} x'_n(\lambda_2 x_2).$$
$$= \lambda_1 \lim_{n \to \infty} x'_n(x_1) + \lambda_2 \lim_{n \to \infty} x'_n(x_2).$$
$$= \lambda_1 x'(x_1) + \lambda_2 x'(x_2) \quad [\text{using ③}].$$

which shows that x' is linear. (12)

x' is bounded and hence Continuous:

Since $\{x'_n\}$ is a Cauchy sequence, so it is bounded ($\because$ every Cauchy sequence is a bounded sequence) Therefore by definition, There exists a Constant $K>0$ such that $\|x'_n\| \leq K$; $\forall n$.

For $x \in X$, we have:

$|x'_n(x)| \leq \|x'_n\| \, \|x\|$ $\quad (\because \|Tx\| \leq \|T\|\|x\|)$

$\leq K\|x\|$; $\forall n$ (by above)

Taking limit as $n \to \infty$, we get:

$|x'(x)| \leq K\|x\|$; $\forall n$ (using (3)).

which shows that x' is bounded and hence Continuous. Hence $x' \in X'$.

To complete the proof, it remains to show that $x'_n \longrightarrow x'$.

By (2), we have:

$|x'_m(x) - x'_n(x)| < \epsilon\|x\|$; $\forall m, n \geq N$.

Since the result holds for every $m \geq N$ and $x'_m(x) \longrightarrow x'(x)$ [by (3)]

we may let $m \to \infty$. Thus letting $\lim\limits_{n\to\infty}$, we get

$|x'(x) - x'_n(x)| \leq \epsilon\|x\|$; $\forall n \geq N$ [by (3)].

$\Rightarrow |(x' - x'_n)(x)| \leq \epsilon\|x\|$; $\forall n \geq N$.

$<$ Changes to $\leq$ because of $m \to \infty$

By taking Sup over all x of norm, we have

$\|x' - x'_n\| \leq \epsilon$; $\forall n \geq N$.

which shows that $\{x_n'\}$ converges to x' i.e. $x_n' \longrightarrow x'$. Consequently X' is complete.

Quotient Spaces:

Definition (2.10): let M be a subspace of a linear space X. We say that any two elements x, y in X are equivalent modulu M if $x - y \in M$ and we write $x \equiv y \bmod M$.

Remark (2.11): It is easy to verify that "equivalence modulu M" is actually an equivalence relation.

ie (i) $x \equiv x \bmod M$ for every x (Reflexive property)
($\because x - x = 0 \in M$ as M is a subspace)

(ii) if $x \equiv y \bmod M$, then $y \equiv x \bmod M$ (symmetric property).
($\because x - y \in M \Rightarrow -(x-y) \in M \Rightarrow y - x \in M$)

(iii) if $x \equiv y \bmod M$ and $y \equiv z \bmod M$, then

$x \equiv z \bmod M$ (Transitive property).
($\because x - y \in M, y - z \in M \Rightarrow x - y + y - z \in M \Rightarrow x - z \in M$)

Furthermore equivalence modulu M can be added and multiplied as if they were ordinary equations.

if $x_1 \equiv x_2 \bmod M$ and $y_1 \equiv y_2 \bmod M$.

Then $x_1 + y_1 = x_2 + y_2 \bmod M$.

and $x_1 y_1 = x_2 y_2 \bmod M$.

$x_1 \equiv x_2 \Rightarrow x_1 - x_2 \in M$
$y_1 \equiv y_2 \Rightarrow y_1 - y_2 \in M$
$(x_1 + y_1) - (x_2 + y_2)$
$= x_1 - x_2 + y_1 - y_2$
$\in M$.

Definition (2.12): let $x \in X$. An equivalence class of x is denoted by $[x]$ and is defined as:

$$[x] = \{y \in X : x \equiv y \bmod M\}.$$

ie $[x] = \{y \in X : x \equiv y \bmod M\}$.

$= \{y \in X : y \equiv x \bmod M\}$ (Symmetry).

$= \{y \in X : y - x \in M\}$.

$= \{y \in X : y - x = m \text{ for some } m \in M\}$.

$= \{y \in X : y = x + m \text{ for some } m \in M\}$.

$= \{x + m : m \in M\}$

$= x + M$.

The collection of all equivalence classes of elements of X is defined as the set :

$$X/M = \{[x] : x \in X\}.$$

Then X/M is a linear space with addition and scaler multiplication is defined by:

$$[x] + [y] = [x+y]$$

and $[\alpha x] = \alpha [x]$.

The linear space X/M is called a Quotient space.

<u>Note</u>: (i) $[x] = [y]$ iff $x - y \in M$.

(ii) If $y \in [x]$, Then $[y] = [x]$ ✓

(iii) The zero element of X/M is $[0]$, which is the same as M.

(iv) $x \in M$ iff $[x]$ is the zero element of X/M.

(v) The quotient norm on X/M is defined by:

$$\|[x]\| = \inf_{y\in[x]} \|y\| .$$
$$= \inf_{m\in M} \|x-m\| .$$

(vi) $\|[x]\| \le \|x\|$.

Theorem (2.13):- Let M be a closed linear subspace in the linear space X. For each $[x] \in X/M$, define
$$\|[x]\| = \inf_{y\in[x]} \|y\| = \inf_{m\in M} \|x-m\| \longrightarrow (1)$$

$y\in[x] \Rightarrow x\equiv y$
$\Rightarrow x-y \in M$
$\Rightarrow x-y = m$
$\Rightarrow x-m = y$

Then $\|[x]\|$ is a norm on X/M ie X/M is a norm linear space.

Proof: To prove that (1) defines norm on X/M, we shall show that for $[x],[y]$ in X/M and α in $\mathbb{K}$, the following conditions are true.

(I) $\|[x]\| \geq 0$. (II) $\|[x]\| = 0$ iff $[x]$ is the zero element of X/M ie $[0] = M$.

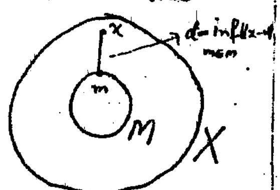
(III) $\|[\alpha x]\| = |\alpha| \, \|[x]\|$ (IV) $\|[x]+[y]\| \le \|[x]\| + \|[y]\|$.

Now clearly by definition:
$$\|[x]\| = \inf_{y\in[x]} \|y\| = \inf_{m\in M} \|x-m\| \geq 0 \quad \text{(using (1))}.$$

Therefore $\|[x]\| \geq 0$, which proves (I).

Now we have by definition (1),

$\|[x]\| = \inf_{m\in M} \|x-m\|$, which shows that $\|[x]\|$ is the distance from x to M.



Since M is closed, so (A set is closed iff [illegible] all of its [illegible])

$\|[x]\| = 0$ iff $x \in M$

iff $[x]$ is the zero element of X/M

ie $[0] = M$. (by note (iv))

which proves (II).

Next for $\alpha \in \mathbb{K}$ and $x \in X$, we have:

$$\|[\alpha x]\| = \inf_{y \in [x]} \|\alpha y\| \qquad (\text{by } ①).$$

$$= |\alpha| \inf_{y \in [x]} \|y\|$$

$$= |\alpha| \, \|[x]\|, \text{ which proves (III).}$$

To prove (IV); For any x, y in X, we have:

$$\|[x] + [y]\| = \|[x+y]\| = \inf_{\substack{u \in [x] \\ v \in [y]}} \|u + v\|$$

$$\leq \inf_{u \in [x]} \|u\| + \inf_{v \in [y]} \|v\|$$

$$= \|[x]\| + \|[y]\|$$

ie $\|[x] + [y]\| \leq \|[x]\| + \|[y]\|$

Since all the conditions of a norm are satisfied. Thus X/M is a norm linear space with norm defined by ①.

Theorem (2-14): Let M be a closed subspace of a Banach space X. Then X/M with the norm defined by ① in Thm (2.13) is also a Banach space.

Proof: Suppose that X is a Banach space ie X is complete as a metric space and we show that X/M is also complete.

If we start with a Cauchy sequence in X/M, then by a Theorem which states, "A Cauchy sequence is convergent iff it has a convergent subsequence", it is enough to show that this sequence has a convergent subsequence. Then by induction it is possible to find a subsequence $\{[x_n]\}$ of the original Cauchy sequence in X/M such that

$$\|[x_1] - [x_2]\| < \frac{1}{2},$$

$$\|[x_2] - [x_3]\| < \frac{1}{2^2}, \text{ ------- , in general}$$

$$\|[x_n] - [x_{n+1}]\| < \frac{1}{2^n} \quad \forall n.$$

we prove that this sequence is convergent in X/M.

Choose $y_1 \in [x_1]$ and select $y_2 \in [x_2]$ such that

$$\|y_1 - y_2\| < \frac{1}{2}.$$

we choose $y_3 \in [x_3]$ such that $\|y_2 - y_3\| < \frac{1}{2^2}$.

Continuing this process, we obtain a sequence $\{y_n\}$ in X, where $y_n \in [x_n]$, such that

$$\|y_n - y_{n+1}\| < \frac{1}{2^n} \text{ for all } n. \longrightarrow ①$$

First we show that $\{y_n\}$ is a Cauchy sequence in X.

If $m < n$, then

$$\|y_m - y_n\| = \|(y_m - y_{m+1}) + (y_{m+1} - y_{m+2}) + \cdots + (y_{n-1} - y_n)\|$$

$$\leq \|y_m - y_{m+1}\| + \|y_{m+1} - y_{m+2}\| + \cdots + \|y_{n-1} - y_n\|.$$

$$< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{n-1}} \quad \text{(by ①)}.$$

$$< \frac{1}{2^{m-1}} \quad \left(\text{Geometric series with common ratio } r = \tfrac{1}{2},\ a = \tfrac{1}{2^m},\ S_m = \frac{a(1-r^n)}{1-r}\right).$$

$$\longrightarrow 0 \text{ as } m \longrightarrow \infty.$$

which shows that $\{y_n\}$ is a cauchy sequence in X. But X is Complete, so $\{y_n\}$ Converges in X.
let $y_n \longrightarrow y \in X$.

Note that $y_n \in [x_n]$, so $[y_n] = [x_n]$ (by Note).

Therefore $\|[x_n] - [y]\| = \|[y_n] - [y]\| = \|[y_n - y]\| \leq \|y_n - y\|$ $(\because \|[x]\| \leq \|x\|)$

ie $\|[x_n] - [y]\| \leq \|y_n - y\|$

$$\longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

ie $[x_n] \longrightarrow [y]$ as $n \longrightarrow \infty$.

This shows that the sequence $\{[x_n]\}$ is Convergent ie the subsequence $\{[x_n]\}$ of the original sequence in X/M is Convergent and thus the original cauchy sequence in X/M is Convergent.
consequently X/M is complete and hence a Banach space.